Invertibility of Toeplitz + Hankel Operators and Singular Integral Operators with Flip. – The case of smooth generating functions

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Abstract

It is well known that a Toeplitz operator is invertible if and only if its symbols admits a canonical Wiener–Hopf factorization, where the factors satisfy certain conditions. A similar result holds also for singular integral operators. More general, the dimension of the kernel and cokernel of Toeplitz or singular integral operators which are Fredholm operators can be expressed in terms of the partial indices $\varkappa_1, \dots, \varkappa_N \in \mathbb{Z}$ of an associated Wiener–Hopf factorization problem.

In this paper we establish corresponding results for Toeplitz + Hankel operators and singular integral operators with flip under the assumption that the generating functions are sufficiently smooth (e.g., Hölder continuous). We are led to a slightly different factorization problem, in which pairs $(\varrho_1, \varkappa_1), \dots, (\varrho_N, \varkappa_N) \in \{-1, 1\} \times \mathbb{Z}$, instead of the partial indices appear. These pairs provide the relevant information about the dimension of the kernel and cokernel and thus answer the invertibility problem.

1 Introduction

Let $L^\infty(\mathbb{T})$ stand for the C*-algebra of all essentially bounded and Lebesgue measurable functions defined on the unit circle $\mathbb{T} = \{ \ z \in \mathbb{C} \ : \ |z| = 1 \ \}$, and let L^2 stand for the Hilbert space of all square integrable functions defined on $\mathbb{T}$. Let H^2 ($\overline{H^2}$, resp.) stand for the Hardy space consisting of all functions $f \in L^2$ for which the Fourier coefficients

$$f_n \ = \ \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \, d\theta \tag{1}$$

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vanish for all $n < 0$ ($n > 0$, resp.). Moreover, let $H^\infty = L^\infty(\mathbb{T}) \cap H^2$ and $\overline{H^\infty} = L^\infty(\mathbb{T}) \cap \overline{H^2}$ be the usual Hardy spaces of essentially bounded function. Note that H^∞ and $\overline{H^\infty}$ are Banach subalgebras of $L^\infty(\mathbb{T})$. Finally, let $C(\mathbb{T})$ stand for the C*-algebra of all continuous functions defined on the unit circle.

Given a Banach space X, let X^N stand for the Banach space of all $N \times 1$ vectors with entries in X, and let $X^{N\times N}$ stand for the Banach space of all $N \times N$ matrices with entries in X. Given a Banach algebra B, we denote by GB the group of all invertible elements in B. A Banach subalgebra B_1 of a Banach algebra B_0 is called inverse closed (in B_0) if $b \in B_1 \cap GB_0$ implies $b \in GB_1$.

For an $N \times N$ matrix valued function $A \in L^\infty(\mathbb{T})^{N\times N}$, the multiplication operator generated by A is defined by

$$M(A) \;\;:\;\; (L^2)^N \to (L^2)^N, \;\; f(e^{i\theta}) \mapsto A(e^{i\theta})f(e^{i\theta}). \tag{2}$$

The Riesz projection P and the associated projection Q acting on $(L^2)^N$ are given by

$$P \;:\; \sum_{n=-\infty}^{\infty} f_n e^{in\theta} \mapsto \sum_{n=0}^{\infty} f_n e^{in\theta}, \qquad Q \;:\; \sum_{n=-\infty}^{\infty} f_n e^{in\theta} \mapsto \sum_{n=-\infty}^{-1} f_n e^{in\theta}. \tag{3}$$

Note that $(H^2)^N$ is the image of the Riesz projection P. The flip operator J is defined by

$$J \;\;:\;\; (L^2)^N \to (L^2)^N, \;\; f(e^{i\theta}) \mapsto e^{-i\theta} f(e^{-i\theta}). \tag{4}$$

It is well known that for $A \in L^\infty(\mathbb{T})^{N\times N}$ the Toeplitz operator

$$T(A) \;\;=\;\; PM(A)P \tag{5}$$

acting on $(H^2)^N$ is a Fredholm operator if and only if A possesses a factorization of the form

$$A(t) \;\;=\;\; A_-(t)\Lambda(t)A_+(t), \qquad t \in \mathbb{T}, \tag{6}$$

where $\Lambda(t) = \operatorname{diag}(t^{\varkappa_1}, \dots, t^{\varkappa_N})$ is a diagonal matrix with $\varkappa_1, \dots, \varkappa_N \in \mathbb{Z}$, and the factors A_+ and A_- satisfy the following conditions:

(i) $A_+ \in (H^2)^{N\times N}$, $A_+^{-1} \in (H^2)^{N\times N}$;

(ii) $A_- \in (\overline{H^2})^{N\times N}$, $A_-^{-1} \in (\overline{H^2})^{N\times N}$;

(iii) The operator $M(A_+^{-1})PM(A_-^{-1})$, which is a well defined mapping from $C(\mathbb{T})^N$ into the Lebesgue space $L^1(\mathbb{T})^N$, can be extented by continuity to a linear bounded operator acting from $(L^2)^N$ into $(L^2)^N$.

The integers $\varkappa_1, \ldots, \varkappa_N$ are called the partial indices of the above factorization and are uniquely determined up to change of order. A necessary (but not sufficient) condition for the Fredholmness of $T(A)$ is that $A \in GL^\infty(\mathbb{T})^{N\times N}$. If $T(A)$ is a Fredholm operator, then the dimension of the kernel and cokernel are given by

$$\dim\ker T(A) = -\sum_{\varkappa_j<0} \varkappa_j, \qquad \dim\ker T(A)^* = \sum_{\varkappa_j>0} \varkappa_j. \tag{7}$$

Here "$*$" stands for the adjoint of an operator. The index of $T(A)$, i.e., the number $\operatorname{ind} T(A) := \dim\ker T(A) - \dim\ker T(A)^*$, is equal to $-\varkappa$, where

$$\varkappa \;=\; \sum_{j=1}^{N} \varkappa_j \tag{8}$$

is the so-called total index of the factorization. In particular, the Toeplitz operator $T(A)$ is invertible if and only if A admits a canonical factorization, i.e., a factorization where all partial indices are zero.

A factorization of a matrix function in the form (6) with the properties (i)–(iii) is sometimes called a generalized factorization or a Φ-factorization in the space L^2. For further information about this type of factorization and generalizations of it, we refer the reader to the monographs [3, 6, 4].

For some classes of functions (e.g., piecewise continuous matrix functions) there exist different Fredholm criteria, which are easier to verify. There also exist explicit formulas for total index. However, in the case $N > 1$, the explicit construction of a factorization, or, at least the determination of the partial indices is often the only possibility to answer the question about the invertibility (and, more general, to calculate the dimension of the kernel and cokernel in the case of Fredholm operators).

For singular integral operators (with $A, B \in L^\infty(\mathbb{T})^{N\times N}$)

$$S(A,B) \;=\; PM(A) + QM(B), \tag{9}$$

which are defined on $(L^2)^N$, a similar result holds. Namely, $S(A,B)$ is Fredholm if and only if $A, B \in G(L^\infty(\mathbb{T})^{N\times N})$ and if $T(AB^{-1})$ is a Fredholm operator. The latter means that the matrix function AB^{-1} admits a factorization of the above kind. We remark in this connection that

$$S(A,B) \;=\; \Big(I + PM(AB^{-1})Q\Big)\Big(T(AB^{-1}) + Q\Big)M(B), \tag{10}$$

where $I + PM(AB^{-1})Q$ and $M(B)$ are invertible operators. Hence the problem of computing the dimension of the kernel and cokernel of a singular integral operator can be reduced to a factorization problem with the determination of partial indices.

Fredholm criteria related to a factorization problem and formulas for the dimension of the kernel and cokernel similar to above have so far not been known for singular integral operators with flip,

$$PM(A) + PJM(B) + QJM(C) + QM(D), \qquad A, B, C, D \in L^\infty(\mathbb{T})^{N\times N}, \tag{11}$$

not even in the case where the generating functions are smooth. Also for Toeplitz + Hankel operators,

$$T(A) + H(B), \qquad A, B \in L^\infty(\mathbb{T})^{N\times N}, \tag{12}$$

such results have not yet been obtained. Here

$$H(B) \;=\; PM(B)JP \tag{13}$$

stands for the Hankel operator acting on $(H^2)^N$ with the generating function $B \in L^\infty(\mathbb{T})^{N\times N}$. Only in a recent paper of E. L. Basor and the author [1] it has been observed that the invertibility of special class of Toeplitz + Hankel operators might be related to a factorization problem.

The Fredholm theory of Toeplitz + Hankel operators with piecewise continuous functions can be found in [7] (see also [2, Sect. 4.95–4.102]). Several aspects of the Fredholm theory of singular integral operators with flip (also in a different settings) can be found in the monograph [5].

We remark that there exists a "classical" trick, which allows to reduce singular integral operators with flip to singular integral operators without flip (and thus to a factorization problem). This trick will be sketched below. Unfortunately, this trick leads only to sufficient conditions and gives in general only estimates on the dimensions of the kernel and cokernel.

The purpose of this paper is to consider general singular integral operators with flip and Toeplitz + Hankel operators with sufficiently smooth (e.g., Hölder continuous) matrix valued generating functions. In the case where these operators are Fredholm we will establish formulas for the dimension of the kernel and cokernel. Note that (in the case of continuous generating functions) Fredholm criteria are easy to obtain. These formulas will rely on a factorization problem, which is slightly different from the classical Wiener–Hopf factorization. Instead of the partial indices $\varkappa_1, \dot{;} \varkappa_N \in \mathbb{Z}$, a collection of pairs $(\varrho_1, \varkappa_1), \dots (\varrho_N, \varkappa_N) \in \{-1, 1\} \times \mathbb{Z}$ appears, which contain the relevant information about the dimension of the kernel and cokernel and allow us to give an answer to the invertibility problem.

The general case, i.e., Fredholm criteria in terms of a factorization problem for singular integral operators with flip and Toeplitz + Hankel operators with generating functions in $L^\infty(\mathbb{T})^{N\times N}$, will be deferred to a future paper.

Let us state some basic relations between the operators introduced above. Obviously, $P^2 = P$, $Q^2 = Q$ and $P + Q = I$ by definition. Moreover,

$$J^2 = I, \qquad JPJ = Q \quad \text{and} \quad JM(A)J = M(\widetilde{A}), \tag{14}$$

where $\widetilde{A}$ stands for the function defined by

$$\widetilde{A}(t) \;=\; A(1/t), \qquad t \in \mathbb{T}. \tag{15}$$

For functions $A, B \in L^\infty(\mathbb{T})^{N\times N}$, the following relation for multiplication operators holds:

$$M(AB) \;=\; M(A)M(B). \tag{16}$$

From this and the above relations, one can deduce well known identities for Toeplitz and Hankel operators:

$$\begin{aligned} T(AB) &= T(A)T(B) + H(A)H(\widetilde{B}), && (17)\\ H(AB) &= T(A)H(B) + H(A)T(\widetilde{B}). && (18) \end{aligned}$$

Now let us explain how the above mentioned "classical" trick works in regard to singular integral operators with flip. It works, of course, also for Toeplitz + Hankel operators. First consider the identity

$$\frac{1}{2}\begin{pmatrix} I & I \\ J & -J \end{pmatrix}\begin{pmatrix} X+YJ & 0 \\ 0 & X-YJ \end{pmatrix}\begin{pmatrix} I & J \\ I & -J \end{pmatrix} \;=\; \begin{pmatrix} X & Y \\ JYJ & JXJ \end{pmatrix}, \tag{19}$$

where X and Y are arbitrary operators acting on $(L^2)^N$. Note that the block operators on the left and the right of the left hand side of the equation are the inverses of each other. Given $a, b, c, d \in L^\infty(\mathbb{T})^{N\times N}$, write

$$A \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L^\infty(\mathbb{T})^{2N\times 2N} \tag{20}$$

and introduce two singular integral operators with flip:

$$\begin{aligned} \Phi(A) &= PM(a) + PM(b)J + QM(\tilde{d}) + QM(\tilde{c})J, && (21)\\ \Phi'(A) &= PM(a) - PM(b)J + QM(\tilde{d}) - QM(\tilde{c})J. && (22) \end{aligned}$$

Notice the slight change in notation in comparison with (11). With $X = PM(a) + QM(\tilde{d})$ and $Y = PM(b) + QM(\tilde{c})$ we can employ (19), and it follows that problem of Fredholmness, invertibility and dimension of the kernel and cokernel are the same for the operators

$$\begin{pmatrix} \Phi_+(A) & 0 \\ 0 & \Phi_-(A) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} PM(a)+QM(\tilde{d}) & PM(b)+QM(\tilde{c}) \\ QM(\tilde{b})+PM(c) & QM(\tilde{a})+PM(d) \end{pmatrix}. \tag{23}$$

However, this last operator can be rewritten as

$$P\begin{pmatrix} M(a) & M(b) \\ M(c) & M(d) \end{pmatrix} + Q\begin{pmatrix} M(\tilde{d}) & M(\tilde{c}) \\ M(\tilde{b}) & M(\tilde{a}) \end{pmatrix} \;=\; PM(A) + QM(W\widetilde{A}W) \tag{24}$$

with a constant $2N \times 2N$ matrix

$$W = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{25}$$

The operator (24) is a usual singular integral operator with generating functions of twice the original matrix size. By what has been said above about singular integral operators, one is led to the factorization of matrix function $AW\widetilde{A}^{-1}W$ in the form (6).

The disadvantage of this trick is that one cannot study $\Phi(A)$ alone, but one is compelled to take also the "conjugate" operator $\Phi'(A)$ into account. In the worst case it can happen that $\Phi(A)$ is a Fredholm operator whereas $\Phi'(A)$ is not, in which case one obtains no information at all about $\Phi(A)$.

2 First results about Toeplitz + Hankel operators

In this section we first establish the basic properties of general Toeplitz + Hankel operators $T(A) + H(B)$ with $A, B \in L^\infty(\mathbb{T})^{N\times N}$. Then we introduce two special classes of such Toeplitz + Hankel operators and consider their basic properties, too. The further study of these particular as well as of the general Toeplitz + Hankel operators will be continued in later sections.

The following necessary condition for the Fredholmness of general Toeplitz + Hankel operators is certainly well known. For completeness sake, we present it with a proof.

Proposition 2.1 *Let $A, B \in L^\infty(\mathbb{T})^{N\times N}$, and assume that $T(A)+H(B)$ is Fredholm. Then $A \in G(L^\infty(\mathbb{T})^{N\times N})$.*

Proof. If $T(A) + H(B)$ is Fredholm, then there exist $\delta > 0$ and a finite rank projection K on the kernel of $T(A) + H(B)$ such that

$$\|T(A)f + H(B)f\|_{(H^2)^N} + \|Kf\|_{(H^2)^N} \;\;\geq\;\; \delta\|f\|_{(H^2)^N}$$

for all $f \in (H^2)^N$. Replacing f by Pf and applying the estimate $\|Pf\| \geq \|f\| - \|Qf\|$, it follows that

$$\|T(A)f + H(B)f\|_{(L^2)^N} + \|KPf\|_{(L^2)^N} + \delta\|Qf\|_{(L^2)^N} \;\;\geq\;\; \delta\|f\|_{(L^2)^N}.$$

for all $f \in (L^2)^N$. Introducing the isometries $U_{\pm n} : (L^2)^N \to (L^2)^N, f(t) \mapsto t^{\pm n}f(t)$ and replacing f by $U_n f$, we obtain

$$\|U_{-n}T(A)U_nf + U_{-n}H(B)U_nf\|_{(L^2)^N} + \|KPU_nf\|_{(L^2)^N} + \delta\|U_{-n}QU_nf\|_{(L^2)^N} \geq \delta\|f\|_{(L^2)^N}.$$

Because $U_{\pm n}$ commute with multiplication operators and $U_nJ = JU_{-n}$, we can write

$$U_{-n}T(A)U_n = U_{-n}PU_nM(A)U_{-n}PU_n, \qquad U_{-n}H(B)U_n = U_{-n}PU_{-n}M(B)JU_{-n}PU_n.$$

Observe that $U_{-n}PU_n \to I$ and $U_{-n}PU_{-n} \to 0$ strongly as $n \to \infty$. Hence it follows that $U_{-n}T(A)U_n \to M(A)$ and $U_{-n}H(B)U_n \to 0$ strongly as $n \to \infty$. Now we take the limit in the above norm estimate. Since $U_n \to 0$ weakly and K is compact, we have $KPU_n \to 0$ strongly. Moreover, $U_{-n}QU_n \to 0$ strongly. We obtain that

$$\|M(A)f\|_{(L^2)^N} \;\; \geq \;\; \delta\|f\|_{(L^2)^N}$$

for all $f \in (L^2)^N$. From this it immediately follows that $A \in G(L^\infty(\mathbb{T})^{N\times N})$. □

For continuous matrix valued functions A and B, the just stated necessary Fredholm condition is also sufficient. Recall in this connection that the winding number of a complex valued nonvanishing continuous functions a defined on the unit circle is given by

$$\operatorname{wind} a \;\; = \;\; \left[\frac{1}{2\pi}\arg a(e^{i\theta})\right]_{\theta=0}^{2\pi}, \tag{26}$$

where the argument $\arg a(e^{i\theta})$ is chosen continuously on $[0, 2\pi]$. Again, the following result is well known, and we present the proof only for completeness sake.

Proposition 2.2 *Let $A, B \in C(\mathbb{T})^{N\times N}$. Then $T(A) + H(B)$ is Fredholm if and only if $A \in G(C(\mathbb{T})^{N\times N})$. Moreover, if this is true, then* $\operatorname{ind}(T(A) + H(B)) = -\operatorname{wind}\det A$.

Proof. It suffices to remark that the Hankel operator with a continuous generating function is compact. Hence, by making use of (17), it is easy to see that a Fredholm regularizer for $T(A)+H(B)$ is given by $T(A^{-1})$. As to the index formula, we remark that for $B \in C(\mathbb{T})^{N\times N}$, $A \in G(C(\mathbb{T})^{N\times N})$,

$$\operatorname{ind}(T(A) + H(B)) \; = \; \operatorname{ind} T(A) \; = \; \operatorname{ind} T(\det A) \; = \; -\operatorname{wind}\det A.$$

The last equality is the well known formula for the Fredholm index of a scalar Toeplitz operator with continuous symbol. For the precise justification of the second last equality see, e.g., [2, Thm. 2.94] □

After these results for general Toeplitz + Hankel operators we are going to consider two special classes of Toeplitz + Hankel operators. These operators possess a number of unexpected properties.

In what follows, let $W \in \mathbb{C}^{N\times N}$ be any matrix such that $W^2 = I$. For $A \in L^\infty(\mathbb{T})^{N\times N}$, we introduce the operators

$$\mathcal{M}_W(A) \;\; = \;\; T(A) + H(AW), \tag{27}$$

$$\mathcal{N}_W(A) \;\; = \;\; T(A) + H(W\widetilde{A}). \tag{28}$$

What makes these classes of operators so interesting for us is the fact that an analogue of formula (17) holds. Indeed,

$$\begin{aligned}
\mathcal{M}_W(AB) &= \mathcal{M}_W(A)\mathcal{M}_W(B) + H(AW)\mathcal{M}_W(W\widetilde{B}W - B), && (29)\\
\mathcal{N}_W(AB) &= \mathcal{N}_W(A)\mathcal{N}_W(B) + \mathcal{N}_W(W\widetilde{A}W - A)H(W\widetilde{B}). && (30)
\end{aligned}$$

These formulas can be verified straightforwardly by using (17), (18), and the assumption that W is a constant matrix with $W^2 = I$:

$$\begin{aligned}
\mathcal{M}_W(AB) &= T(AB) + H(ABW)\\
&= T(A)T(B) + H(A)H(\widetilde{B}) + T(A)H(BW) + H(A)T(\widetilde{B}W)\\
&= T(A)\mathcal{M}_W(B) + H(A)\mathcal{M}_W(\widetilde{B}W)\\
&= T(A)\mathcal{M}_W(B) + H(AW)\mathcal{M}_W(W\widetilde{B}W)\\
&= \mathcal{M}_W(A)\mathcal{M}_W(B) + H(AW)\mathcal{M}_W(W\widetilde{B}W - B).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{N}_W(AB) &= T(AB) + H(W\widetilde{A}\widetilde{B})\\
&= T(A)T(B) + H(A)H(\widetilde{B}) + T(W\widetilde{A})H(\widetilde{B}) + H(W\widetilde{A})T(B)\\
&= \mathcal{N}_W(A)T(B) + \mathcal{N}_W(W\widetilde{A})H(\widetilde{B})\\
&= \mathcal{N}_W(A)T(B) + \mathcal{N}_W(W\widetilde{A}W)H(W\widetilde{B})\\
&= \mathcal{N}_W(A)\mathcal{N}_W(B) + \mathcal{N}_W(W\widetilde{A}W - A)H(W\widetilde{B}).
\end{aligned}$$

Next we introduce the set

$$(L^\infty)_W^{N\times N} \;=\; \Big\{\, A \in L^\infty(\mathbb{T})^{N\times N} \;:\; W\widetilde{A}W = A \,\Big\}. \qquad (31)$$

We remark that $(L^\infty)_W^{N\times N}$ is an inverse closed Banach subalgebra of $L^\infty(\mathbb{T})^{N\times N}$.

Under additional assumptions on the functions A or B, formulas (29) and (30) can be simplied. Indeed,

$$\begin{aligned}
\mathcal{M}_W(AB) &= \mathcal{M}_W(A)\mathcal{M}_W(B) && \text{if } A \in (\overline{H^\infty})^{N\times N} \text{ or } B \in (L^\infty)_W^{N\times N}; && (32)\\
\mathcal{N}_W(AB) &= \mathcal{N}_W(A)\mathcal{N}_W(B) && \text{if } A \in (L^\infty)_W^{N\times N} \text{ or } B \in (H^\infty)^{N\times N}. && (33)
\end{aligned}$$

Consequently, in some cases the mappings $A \mapsto \mathcal{M}_W(A)$ and $A \mapsto \mathcal{N}_W(A)$ are multiplicative. This is halfway not surprising. In fact,

$$\begin{aligned}
\mathcal{M}_W(A) &= T(A) && \text{if } A \in (\overline{H^\infty})^{N\times N}; && (34)\\
\mathcal{N}_W(A) &= T(A) && \text{if } A \in (H^\infty)^{N\times N}. && (35)
\end{aligned}$$

Hence in these cases we are dealing just with usual Toeplitz operators which have symbols in $(\overline{H^\infty})^{N\times N}$ and $(H^\infty)^{N\times N}$, respectively.

More interesting is the case where $A \in (L^\infty)_W^{N\times N}$. It turns out that then both of the above types of operators coincide:

$$\mathcal{M}_W(A) \;=\; \mathcal{N}_W(A) \qquad \text{if } A \in (L^\infty)_W^{N\times N}. \tag{36}$$

Moreover, the following result shows that both the invertibility and the Fredholm problem can be solved completely in a very simple way.

Corollary 2.3 *Let $A \in (L^\infty)_W^{N\times N}$. Then the following is equivalent:*

(i) $A \in G(L^\infty)_W^{N\times N}$;

(ii) $\mathcal{M}_W(A) = \mathcal{N}_W(A)$ is invertible;

(iii) $\mathcal{M}_W(A) = \mathcal{N}_W(A)$ is Fredholm.

If this is fulfilled, then the inverse of $\mathcal{M}_W(A) = \mathcal{N}_W(A)$ is given by $\mathcal{M}_W(A^{-1}) = \mathcal{N}_W(A^{-1})$.

Proof. Because of the multiplicative relations (32) or (33), it follows that (i) implies (ii), where the inverse of $\mathcal{M}_W^{-1}(A) = \mathcal{N}_W^{-1}(A)$ is given by $\mathcal{M}_W(A^{-1}) = \mathcal{N}_W(A^{-1})$. The implication (ii)⇒(iii) is obvious. The fact that (iii) implies (i) follows from Proposition 2.1 in connection with the inverse closedness of $(L^\infty)_W^{N\times N}$ in $L^\infty(\mathbb{T})^{N\times N}$. □

The operators $\mathcal{M}_W(A)$ and $\mathcal{N}_W(A)$ are not completely unrelated with each other. First of all, there is a connection by means of the adjoints,

$$(\mathcal{M}_W(A))^* \;=\; \mathcal{N}_{W^*}(A^*) \quad \text{or} \quad (\mathcal{N}_W(A))^* \;=\; \mathcal{M}_{W^*}(A^*), \tag{37}$$

where $A^*(t) := (A(t))^*$. Here we need only remark that $P^* = P$, $J^* = J$ and $M(A)^* = M(A^*)$, from which $T(A)^* = T(A^*)$ and $H(A)^* = H(\widetilde{A}^*)$ follows.

Another relation is established by the identity

$$\mathcal{M}_W(A)\mathcal{N}_W(B) \;=\; T(AB) + H(AW\widetilde{B}) \tag{38}$$

Indeed,

$$\begin{aligned}
\mathcal{M}_W(A)\mathcal{N}_W(B) &= \Big(T(A) + H(AW)\Big)\Big(T(B) + H(W\widetilde{B})\Big)\\
&= T(A)T(B) + H(A)H(\widetilde{B}) + H(AW)T(B) + T(AW)H(\widetilde{B})\\
&= T(AB) + H(AW\widetilde{B}).
\end{aligned}$$

Here we have only used the assumption that W is a constant matrix with $W^2 = I$ and formulas (17) and (18).

Finally, we illustrate some further interesting consequences of the relations (32) and (33).

Corollary 2.4 *Let $A \in L^\infty(\mathbb{T})^{N\times N}$.*

(i) If A admits a factorization $A(t) = A_-(t)A_0(t)$ with $A_- \in G(\overline{H^\infty})^{N\times N}$ and $A_0 \in G(L^\infty)_W^{N\times N}$, then $\mathcal{M}_W(A)$ is invertible and the inverse equals $\mathcal{M}_W(A_0^{-1})T(A_-^{-1})$.

(ii) If A admits a factorization $A(t) = A_0(t)A_+(t)$ with $A_0 \in G(L^\infty)_W^{N\times N}$ and $A_+ \in G(H^\infty)^{N\times N}$, then $\mathcal{N}_W(A)$ is invertible and the inverse equals $T(A_+^{-1})\mathcal{N}_W(A_0^{-1})$.

Proof. As to assertion (i), it follows from (32) and (34) that $\mathcal{M}_W(A) = T(A_-)\mathcal{M}_W(A_0)$. The inverse of $T(A_-)$ equals $T(A_-^{-1})$ and the inverse of $\mathcal{M}_W(A_0)$ equals $\mathcal{M}_W(A_0^{-1})$. In regard to assertion (ii), we use (33) and (35) and obtain $\mathcal{N}_W(A) = T(A_+)\mathcal{N}_W(A_0)$. The inverse of $T(A_-)$ equals $T(A_-^{-1})$ and the inverse of $\mathcal{N}_W(A)$ equals $\mathcal{N}_W(A_0^{-1})$. $\Box$

We conclude this section by making some more or less heuristic remarks, which should serve as a motivation for the kinds of factorizations that we are going to consider in the following section.

The above necessary condition for the invertibility of $\mathcal{M}_W(A)$ (and likewise for $\mathcal{N}_W(A)$) is certainly for several reason far away from being sufficient. In analogy to the usual theory of the Wiener–Hopf factorization one may guess that under certain conditions there exist a factorization of the form

$$A(t) \;=\; A_-(t)R(t)A_0(t) \tag{39}$$

with appropriate conditions on the factors A_-, A_0, and where the middle factor R is of a particularly simple form. Indeed, if $A_- \in G(\overline{H^\infty})^{N\times N}$ and $A_0 \in G(L^\infty)_W^{N\times N}$, then the invertibility of $\mathcal{M}_W(A)$ is equivalent to the invertibility of $\mathcal{M}_W(R)$. More general, the dimensions of the kernel and cokernel of $\mathcal{M}_W(A)$ coincide with those for $\mathcal{M}_W(R)$. If R is of a particularly simple form, then one can hope that these dimensions can be calculated.

It turns out that the factorization of the form (39), which may deserve the name "asymmetric", can be related to some kind of Wiener–Hopf factorization, which looks kind of "antisymmetric". Indeed, if we are given the factorization (39), then

$$AW\widetilde{A}^{-1} \;=\; A_-R(t)A_0W\widetilde{A}_0^{-1}\widetilde{R}^{-1}\widetilde{A}_-^{-1} \;=\; A_-RW\widetilde{R}^{-1}\widetilde{A}_-^{-1}, \tag{40}$$

where the last equality follows from the presumed property $A_0 = W\widetilde{A}_0W$ of the factor A_0. Replacing the product $R(t)W\widetilde{R}^{-1}(t)$ by the notation $D(t)$, we arrive at a factorization

$$F(t) \;=\; A_-(t)D(t)\widetilde{A}_-^{-1}(t) \tag{41}$$

of the matrix function

$$F(t) \;=\; A(t)W\widetilde{A}^{-1}(t). \tag{42}$$

Assuming for a moment that $D(t)$ is of an appropriate form, it follows that (41) is some kind of Wiener–Hopf factorization, where the right and the left factors are related with each other in an "antisymmetric" way.

Similarly, the analysis of the operator $\mathcal{N}_W(A)$ may lead to a factorization of the form

$$A(t) \;=\; A_0(t)R(t)A_+(t), \tag{43}$$

again with suitable conditions on the factors. Elaborating on this "asymmetric" factorization, we arrive at the following "antisymmetric" factorization,

$$G(t) \;=\; \widetilde{A}_+^{-1}(t)D(t)A_+(t) \tag{44}$$

of the matrix function

$$G(t) \;=\; \widetilde{A}^{-1}(t)WA(t). \tag{45}$$

Here $D(t)$ stands for $\widetilde{R}^{-1}(R)WR(t)$, which slightly differs from the previous situation.

The reader should observe that whereas the "asymmetric" factorizations (39) and (43) are of different types, the "antisymmetric" factorizations (41) and (44) is essentially of the same form only that different notation has been used.

3 Some results about factorizations

3.1 The usual factorization within a Banach algebra

Throughout the rest of this paper let $\mathcal{B}$ stand for a Banach algebra of functions defined on the unit circle such that the following properties are fulfilled:

(a) $\mathcal{B}$ is an inverse closed Banach subalgebra of $C(\mathbb{T})$;

(b) $\mathcal{B}$ contains all trigonometric polynomials;

(c) If $a \in \mathcal{B}$, then $\widetilde{a} \in \mathcal{B}$;

(d) For each N, each matrix function $A \in G\mathcal{B}^{N\times N}$ admits a factorization of the form

$$A(t) \;=\; A_-(t)\Lambda(t)A_+(t) \tag{46}$$

where $\Lambda(t) = \operatorname{diag}(t^{\varkappa_1}, t^{\varkappa_2}, \ldots, t^{\varkappa_N})$ with $\varkappa_1, \ldots, \varkappa_N \in \mathbb{Z}$,

$$A_+ \in G\mathcal{B}_+^{N\times N} \quad \text{and} \quad A_- \in G\mathcal{B}_-^{N\times N}, \tag{47}$$

where

$$\mathcal{B}_+ \;:=\; \mathcal{B}\cap H^\infty \quad \text{and} \quad \mathcal{B}_- \;:=\; \mathcal{B}\cap \overline{H^\infty}. \tag{48}$$

Examples of Banach algebras $\mathcal{B}$ having the properties (a)–(d) are the Wiener algebra W or the Banach algebras C^α of all Hölder continuous functions defined on the unit circle with exponents $0 < \alpha < 1$.

From the factorization point of view, only the assumptions (b) and (d) and the condition that $\mathcal{B}$ is a Banach subalgebra of $L^\infty(\mathbb{T})$ are important. More specifically, one refers to the factorization (46) with the properties (47) as a factorization within the Banach algebra $\mathcal{B}$. Related to this concept are such notions as that of decomposing Banach algebras and Banach algebras with factorization property. We will go into these details, but simply refer the reader to [2, Sect. 10.14–10.23]. We also note that $\mathcal{B}_+$ and $\mathcal{B}_-$ defined in (48) are Banach subalgebras of $\mathcal{B}$ containing the unit element.

It is obvious that a factorization in such a Banach algebra is automatically a generalized factorization (or, Φ-factorization) in the space L^2. In particular, $A_+ \in G(H^\infty)^{N\times N}$ and $A_- \in G(\overline{H^\infty})^{N\times N}$, and thus the factors A_+ and A_- satisfy the conditions (i)–(iii) stated in the introduction.

Our assumption (a) is motivated by the circumstance that we will confine ourselves to continuous matrix valued functions because in this case Fredholm criteria for Toeplitz + Hankel operators and singular integral operators with flip are easy to obtain. The inverse closedness is needed for the conclusion that each function $A \in \mathcal{B}^{N\times N}$ which is invertible (in $C(\mathbb{T})^{N\times N}$) admits a factorization of the above kind.

The assumption (c) will be important for the definition of another type of factorization that we will introduced later on. We remark in this connection the obvious fact that $A \in \mathcal{B}_+^{N\times N}$ if and only if $\widetilde{A} \in \mathcal{B}_-^{N\times N}$. Consequently, $A \in G\mathcal{B}_+^{N\times N}$ if and only if $\widetilde{A} \in G\mathcal{B}_-^{N\times N}$.

As has already been noted in the introduction, the partial indices of such factorizations are uniquely determined up to change of order. In fact, the order of the partial indices can be changed in any desired way. Namely, one can replace $F_-(t)$ with $F_-(t)\Pi^{-1}$, $\Lambda(t)$ with $\Pi\Lambda(t)\Pi^{-1}$ and $F_+(t)$ with $\Pi F_+(t)$, where Π is a suitable permutation matrix.

The following result is well known [3, 6] and answers the question about the uniqueness of the factors A_+ and A_- in a factorization. In order to simplify the statement we will assumes without loss of generality that the partial indices are ordered increasingly. Then the factors corresponding to different factorizations are related with each other by certain rational block triangular matrix functions whose structure is determined by the multiple occurrence of same values for the partial indices. In this regard, we introduce the notation I_l for the identity matrix of size $l \times l$.

Proposition 3.1 *Assume that we are given two factorizations of a function $F \in G\mathcal{B}^{N\times N}$,*

$$F(t) \;=\; F_-^{(1)}(t)\Lambda(t)F_+^{(1)}(t) \;=\; F_-^{(2)}(t)\Lambda(t)F_+^{(2)}(t) \tag{49}$$

with $F_-^{(j)} \in G\mathcal{B}_-^{N\times N}$, $F_+^{(j)} \in G\mathcal{B}_+^{N\times N}$, and

$$\Lambda(t) \;=\; \operatorname{diag}\,(t^{\overline{\varkappa}_1} I_{l_1}, t^{\overline{\varkappa}_2} I_{l_2}, \ldots, t^{\overline{\varkappa}_R} I_{l_R}), \tag{50}$$

where $R \in \{1,2,\ldots\}$, $l_1,\ldots,l_R \in \{1,2,\ldots\}$, $l_1+\ldots+l_R = N$, $\overline{\varkappa}_1,\ldots,\overline{\varkappa}_R \in \mathbb{Z}$ *and*

$$\overline{\varkappa}_1 \;<\; \overline{\varkappa}_2 \;<\ldots<\; \overline{\varkappa}_{N-1} \;<\; \overline{\varkappa}_R. \tag{51}$$

Then there exist matrix functions U *and* V *which are of the form*

$$U(t) = \begin{pmatrix} A_{11} & U_{12}(t) & \cdots & U_{1R}(t) \\ 0 & A_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & U_{R-1,R}(t) \\ 0 & \cdots & 0 & A_{RR} \end{pmatrix}, \quad V(t) = \begin{pmatrix} A_{11} & V_{12}(t) & \cdots & V_{1R}(t) \\ 0 & A_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & V_{R-1,R}(t) \\ 0 & \cdots & 0 & A_{RR} \end{pmatrix} \tag{52}$$

with $A_{jj} \in G\mathbb{C}^{l_j\times l_j}$ *and*

$$U_{jk}(t) \;=\; \sum_{m=0}^{\overline{\varkappa}_k-\overline{\varkappa}_j} A_{jk}^{(m)} t^m, \quad V_{jk}(t) \;=\; t^{\overline{\varkappa}_j-\overline{\varkappa}_k} U_{jk}(t), \quad A_{jk}^{(m)} \in \mathbb{C}^{l_j\times l_k} \tag{53}$$

for $1 \le j < k \le R$ *such that*

$$F_-^{(2)}(t) \;=\; F_-^{(1)}(t)V(t), \qquad F_+^{(1)}(t) \;=\; U(t)F_+^{(2)}(t). \tag{54}$$

Due to the assumption (b) on $\mathcal{B}$, it is not hard too see that $U \in G\mathcal{B}_+^{N\times N}$ and $V \in G\mathcal{B}_-^{N\times N}$. The previous proposition holds, by the way, not only for factorizations within the Banach algebra $\mathcal{B}$, but also for generalized factorizations (see (6) and (i)–(iii)). However, we will not make use of this fact.

Actually, the statement of this proposition can be reversed. If we are given a factorization $F(t) = F_-^{(1)}(t)\Lambda(t)F_+^{(1)}(t)$, introduce functions U and V with the above properties and define $F_-^{(2)}$ and $F_+^{(2)}$ by (54), then also $F(t) = F_-^{(2)}(t)\Lambda(t)F_+^{(2)}(t)$ is such a factorization.

3.2 Antisymmetric factorization within a Banach algebra

In what follows we are going to introduce and study a slightly different type of factorization. It is essentially also a factorization of the form (46), but we require in addition that the factors F_+ and F_- are related with each other by $F_+(t) = \widetilde{F}_-^{-1}(t)$. Moreover, the middle factor is allowed to be of a more general form. Namely,

$$D(t) \;=\; \operatorname{diag}\left(\varrho_1 t^{\varkappa_1}, \varrho_2 t^{\varkappa_2}, \ldots, \varrho_N t^{\varkappa_N}\right) \tag{55}$$

with $\varkappa_1,\ldots,\varkappa_N \in \mathbb{Z}$ and $\varrho_1,\ldots,\varrho_N \in \{-1,1\}$.

More specifically, we are going to consider a factorization of a function $F \in G\mathcal{B}^{N\times N}$ in the form

$$F(t) \;=\; F_-(t)D(t)\widetilde{F}_-^{-1}(t) \quad \text{with} \quad F_- \in G\mathcal{B}_-^{N\times N}, \tag{56}$$

where $D(t)$ is given by (55). Such a factorization will be called an *antisymmetric factorization of F within the Banach algebra $\mathcal{B}$.*

It will turn out that the collection of pairs

$$(\varrho_1, \varkappa_1),\ (\varrho_2, \varkappa_2),\ \ldots\ (\varrho_N, \varkappa_N)\ \in\ \{-1,1\} \times \mathbb{Z} \tag{57}$$

plays the same important role as the collection of the partial indices $\varkappa_1, \ldots, \varkappa_N \in \mathbb{Z}$ in the classical situation. Therefore, we will call this collection the *characteristic pairs* of the antisymmetric factorization of F.

We first study the existence of an antisymmetric factorization. Because $\widetilde{D}^{-1}(t) = D(t)$ for each middle factors of the above kind, it is easy to see that the condition

$$F(t) \;=\; \widetilde{F}^{-1}(t) \tag{58}$$

is necessary for the existence of an antisymmetric factorization of a function F. The following theorem shows that, essentially, this condition is also sufficient.

Theorem 3.2 *Assume that $F \in G\mathcal{B}^{N\times N}$ satisfies the condition $F(t) = \widetilde{F}^{-1}(t)$. Then there exists a function $F_- \in G\mathcal{B}_-^{N\times N}$ such that F can be factored in the form*

$$F(t) \;=\; F_-(t)D(t)\widetilde{F}_-^{-1}(t), \tag{59}$$

where $D(t)$ is given by (55) with certain characteristic pairs (57).

Proof. Because of the assumptions on the Banach algebra $\mathcal{B}$ there exists a factorization

$$F(t) \;=\; F_-(t)\Lambda(t)F_+(t) \tag{60}$$

with $F_\pm \in G\mathcal{B}_\pm^{N\times N}$, $\Lambda(t) = \operatorname{diag}(t^{\varkappa_1}, \ldots, t^{\varkappa_N})$, $\varkappa_1, \ldots, \varkappa_N \in \mathbb{Z}$ and $\varkappa_1 \le \ldots \le \varkappa_N$ without loss of generality. Taking the inverse and replacing t by $1/t$, it follows that

$$\widetilde{F}^{-1}(t) \;=\; \widetilde{F}_+^{-1}(t)\Lambda(t)\widetilde{F}_-^{-1}(t). \tag{61}$$

Because $\widetilde{F}^{-1} = F$, the expressions (60) and (61) are equal and represent factorizations of the form (46). We can apply Proposition 3.1 and write $\Lambda(t)$ in the form (50) with the conditions on parameters R, $l_1, \ldots, l_R$ and $\overline{\varkappa}_1, \ldots, \overline{\varkappa}_N$ stated there. We conclude that there exists a matrix function $U(t)$ of the form (52) such that

$$F_+(t) \;=\; U(t)\widetilde{F}_-^{-1}(t), \tag{62}$$

Combining (62) with (60) and introducing $X(t) = \Lambda(t)U(t)$, it follows that

$$F(t) \;=\; F_-(t)X(t)\widetilde{F}_-^{-1}(t), \tag{63}$$

where $X(t)$ is of the form

$$X(t) = \begin{pmatrix} X_{11}(t) & X_{12}(t) & \cdots & X_{1R}(t) \\ 0 & X_{22}(t) & \ddots & \vdots \\ \vdots & \ddots & \ddots & X_{R-1,R}(t) \\ 0 & \cdots & 0 & X_{RR}(t) \end{pmatrix} \tag{64}$$

with

$$\begin{aligned} X_{jk}(t) &= \sum_{m=\overline{\varkappa}_j}^{\overline{\varkappa}_k} X_{jk}^{(m)} t^m, \qquad X_{jk}^{(m)} \in \mathbb{C}^{l_j \times l_k}, \qquad 1 \le j < k \le R, \\ X_{jj}(t) &= X_j t^{\overline{\varkappa}_j}, \qquad\qquad X_j \in G\mathbb{C}^{l_j \times l_j}, \qquad 1 \le j \le R. \end{aligned}$$

Introduce

$$X_0(t) = \operatorname{diag}\left(X_{11}(t), X_{22}(t), \ldots, X_{RR}(t)\right), \tag{65}$$

write

$$X(t) = (I + N_1(t))X_0(t) = X_0(t)(I + N_2(t)), \tag{66}$$

and observe that $N_1(t)$ and $N_2(t)$ are nilpotent matrix functions. Note also that $N_1 \in \mathcal{B}_-^{N\times N}$ and $N_2 \in \mathcal{B}_+^{N\times N}$. From formula (66) we obtain that $N_1(t)X_0(t) = X_0(t)N_2(t)$, and moreover $N_1^m(t)X_0(t) = X_0(t)N_2^m(t)$ for each m by induction. The matrix functions $(I+N_1(t))^{1/2}$ and $(I + N_2(t))^{1/2}$ are well defined by a series expansion, which is finite due to the nilpotency. Using this series expansion, it follows that

$$(I + N_1(t))^{1/2}X_0(t) = X_0(t)(I + N_2(t))^{1/2}.$$

This in connection with (66) implies

$$X(t) = (I + N_1(t))^{1/2}X_0(t)(I + N_2(t))^{1/2}. \tag{67}$$

From (63) and the assumption $\widetilde{F}^{-1}(t) = F(t)$, it follows that $\widetilde{X}^{-1}(t) = X(t)$. From the representation (64) and the definition (65), we further obtain $\widetilde{X}_0^{-1}(t) = X_0(t)$. On account of (66), it now follows that

$$I = \widetilde{X}(t)X(t) = (I + \widetilde{N}_1(t))\widetilde{X}_0(t)X_0(t)(I + N_2(t)) = (I + \widetilde{N}_1(t))(I + N_2(t)).$$

Hence $I + N_2(t) = (I + \widetilde{N}_1(t))^{-1}$, and, consequently, $(I + N_2(t))^{1/2} = (I + \widetilde{N}_1(t))^{-1/2}$. This in connection with (67) implies that

$$X(t) = (I + N_1(t))^{1/2}X_0(t)(I + \widetilde{N}_1(t))^{-1/2}. \tag{68}$$

From $\widetilde{X}_0^{-1}(t) = X_0(t)$ it follows (by putting $t = 1$ and $t = -1$) that

$$(X_0(1))^2 \ = \ (X_0(-1))^2 \ = \ I.$$

Hence $X_j^2 = I$ for each $1 \le j \le R$. Thus we can write $X_j = T_j \operatorname{diag}(I_{p_j}, -I_{q_j})\, T_j^{-1}$ with certain $T_j \in G\mathbb{C}^{l_j \times l_j}$ and $p_j, q_j \in \{0, 1, \ldots\}$ such that $p_j + q_j = l_j$. It follows that

$$X_0(t) \ = \ T \operatorname{diag}(\varrho_1 t^{\varkappa_1}, \varrho_2 t^{\varkappa_2}, \ldots, \varrho_N t^{\varkappa_N})\, T^{-1} \tag{69}$$

with certain $\varrho_1, \ldots, \varrho_N \in \{-1, 1\}$ where $T = \operatorname{diag}(T_1, T_2, \ldots, T_R) \in G\mathbb{C}^{N\times N}$. Denoting by $D(t)$ the diagonal matrix in (69), it follows in connection with (63) and (68) that

$$F(t) \ = \ F_-(t)(I + N_1(t))^{1/2}\, T\, D(t)\, T^{-1}\, (I + \widetilde{N}_1(t))^{-1/2} \widetilde{F}_-^{-1}(t).$$

Because $(I + N_1(t))^{1/2} \in G\mathcal{B}_-^{N\times N}$, we may replace the expression $F_-(t)(I + N_1(t))^{1/2}\, T$ by the notation $F_-(t)$. In this way, we arrive at the desired factorization (59). □

We remark that an antisymmetric factorization (59) is obviously an antisymmetric factorization of the form

$$F(t) \ = \ \widetilde{F}_+^{-1} D(t) F_+(t) \tag{70}$$

with $F_+ \in G\mathcal{B}_+^{N\times N}$ and the same middle factor $D(t)$. The only difference is that of the notation of the factors. Indeed, $F_+(t) = \widetilde{F}_-^{-1}(t)$ shows the relation.

The next theorem concerns the uniqueness of the characteristic pairs of an antisymmetric factorization up to change of order. Notice first that it is possible to rearrange the order of these pairs in any desired way. Indeed, one can replace $F_-(t)$ with $F_-(t)\Pi^{-1}$ and $D(t)$ with $\Pi D(t) \Pi^{-1}$, where Π is a suitable permutation matrix. The important point is that the replacement of $F_-(t)$ with $F_-(t)\Pi^{-1}$ implies the replacement of $\widetilde{F}_-^{-1}(t)$ with $\Pi \widetilde{F}_-^{-1}(t)$, which fits with the factorization formula (56).

Theorem 3.3 *In an antisymmetric factorization of a function $F \in G\mathcal{B}^{N\times N}$, the characteristic pairs are uniquely determined up to change of order.*

Proof. Because an antisymmetric factorization is automatically also a usual factorization in the Banach algebra $\mathcal{B}$ (except for the slightly different middle factor, which is irrelevant at this place), it follows that the numbers $\varkappa_1, \ldots, \varkappa_N$ are uniquely determined up to change of order. Because the order of the characteristic pairs in an antisymmetric factorization can be rearranged in any desired way, we can assume without loss of generality that $\varkappa_1 \le \ldots \le \varkappa_N$.

Now suppose that we are given two antisymmetric factorizations of F,

$$F(t) \ = \ F_-^{(1)}(t) D^{(1)}(t) (\widetilde{F}_-^{(1)})^{-1}(t) \ = \ F_-^{(2)}(t) D^{(2)}(t) (\widetilde{F}_-^{(2)})^{-1}(t),$$

where $D^{(1)}(t)$ and $D^{(2)}(t)$ are both of the form (55) but with the pairs

$$(\varrho_1^{(1)}, \varkappa_1), \ldots, (\varrho_N^{(1)}, \varkappa_N) \quad \text{and} \quad (\varrho_1^{(2)}, \varkappa_1), \ldots, (\varrho_N^{(2)}, \varkappa_N),$$

respectively. Introducing the parameters $l_1, \ldots, l_R \in \{1, 2, \ldots\}$ and the integers $\overline{\varkappa}_1, \ldots, \overline{\varkappa}_R$ as in Proposition 3.1, we can write

$$\begin{aligned} D^{(1)}(t) &= \operatorname{diag}(S_1^{(1)} t^{\overline{\varkappa}_1}, S_2^{(1)} t^{\overline{\varkappa}_2}, \ldots, S_R^{(1)} t^{\overline{\varkappa}_R}), \\ D^{(2)}(t) &= \operatorname{diag}(S_1^{(2)} t^{\overline{\varkappa}_1}, S_2^{(2)} t^{\overline{\varkappa}_2}, \ldots, S_R^{(2)} t^{\overline{\varkappa}_R}), \end{aligned}$$

where $S_k^{(1)}$ and $S_k^{(2)}$ are diagonal matrices of size $l_k \times l_k$ with entries -1 or 1 on the diagonal. Moreover, we can write $D^{(1)}(t) = \Lambda(t) S^{(1)}$ and $D^{(2)}(t) = \Lambda(t) S^{(2)}$, where $\Lambda(t)$ is of the form (50) and $S^{(j)} = \operatorname{diag}(S_1^{(j)}, S_2^{(j)}, \ldots, S_R^{(j)})$. It follows that

$$F(t) = F_-^{(1)}(t)\Lambda(t)\Big(S^{(1)}(\widetilde{F}_-^{(1)})^{-1}(t)\Big) = F_-^{(2)}(t)\Lambda(t)\Big(S^{(2)}(\widetilde{F}_-^{(2)})^{-1}(t)\Big),$$

are two factorizations of the form (46). We apply Proposition 3.1 and see that

$$F_-^{(2)}(t) = F_-^{(1)}(t)V(t), \qquad S^{(1)}(\widetilde{F}_-^{(1)})^{-1}(t) = U(t)S^{(2)}(\widetilde{F}_-^{(2)})^{-1}(t),$$

where U and V are of the form (52). The last equation can be rewritten as $F_-^{(1)}(t)(S^{(1)})^{-1} = F_-^{(2)}(t)(S^{(2)})^{-1}\widetilde{U}^{-1}(t)$. Combined with the first equation, it follows that

$$S^{(1)} = \widetilde{U}(t) S^{(2)} V^{-1}(t).$$

Because of the block triangular structure of U and V with invertible constant matrices A_k on the block diagonal, we obtain that $S_k^{(1)} = A_k S^{(2)} A_k^{-1}$ for each $k = 1, \ldots, R$. Hence $S_k^{(1)} \sim S_k^{(2)}$, and, consequently, the numbers of 1's and -1's, respectively, on the diagonal of $S_k^{(1)}$ and $S_k^{(2)}$ is the same. From this it follows that the collection of the pairs $(\varrho_k^{(1)}, \varkappa_k)$ is the same as the collection of the pairs $(\varrho_k^{(2)}, \varkappa_k)$ up to change of order. $\Box$

It is possible (similar as has been done in Proposition 3.1) to state the relation between the factors F_- of two different antisymmetric factorizations of a given function. We will omit this result because it is a little bit difficult to state and will not be needed for our purposes.

For a given antisymmetric factorization of a function F with characteristic pairs (57), we introduce the following nonnegative integers:

$\alpha =$ number of $k \in \{1, \ldots, N\}$ for which $\varrho_k = 1$ and $\varkappa_k$ is even;

$\beta =$ number of $k \in \{1, \ldots, N\}$ for which $\varrho_k = 1$ and $\varkappa_k$ is odd;

$\gamma =$ number of $k \in \{1, \ldots, N\}$ for which $\varrho_k = -1$ and $\varkappa_k$ is odd;

$\delta =$ number of $k \in \{1, \dots, N\}$ for which $\varrho_k = -1$ and $\varkappa_k$ is even.

Besides the obvious fact that $\alpha + \beta + \gamma + \delta = N$, the following "a priori" characterization of these numbers can be obtained.

Proposition 3.4 *Assume that $F \in G\mathcal{B}^{N\times N}$ admits an antisymmetric factorization with the numbers $\alpha, \beta, \gamma, \delta$ be defined as above. Then*

$$F(1) \sim \operatorname{diag}(I_{\alpha+\beta}, -I_{\gamma+\delta}) \quad \textit{and} \quad F(-1) \sim \operatorname{diag}(I_{\alpha+\gamma}, -I_{\beta+\delta}). \tag{71}$$

Proof. Putting $t = 1$ or $t = -1$ in the factorization $F(t) = F_-(t)D(t)\widetilde{F}_-^{-1}(t)$ it follows that $F(1) \sim D(1)$ and $F(-1) \sim D(-1)$. Now the assertion follows from the facts that $D(1) \sim \operatorname{diag}(I_{\alpha+\beta}, -I_{\gamma+\delta})$ and $D(-1) \sim \operatorname{diag}(I_{\alpha+\gamma}, -I_{\beta+\delta})$ as can easily be seen. $\Box$

In regard to the previous proposition, we remark that the necessary condition $F(t) = \widetilde{F}^{-1}(t)$ for the existence of an antisymmetric factorization of $F \in G\mathcal{B}^{N\times N}$ implies $F(1)^2 = F(-1)^2 = I$ by just putting $t = 1$ or $t = -1$. Hence for given F (and thus given $F(1)$ and $F(-1)$), the values of

$$\alpha + \beta, \qquad \gamma + \delta, \qquad \alpha + \gamma, \qquad \beta + \delta. \tag{72}$$

can immediately be determined without knowing the antisymmetric factorization of F or the corresponding characteristic pairs:

3.3 Asymmetric factorizations within a Banach algebra

In regard to the discussion at the end of Section 2 we are going to show that each $A \in G\mathcal{B}^{N\times N}$ can be factored in certain "asymmetric" ways.

As a first auxiliary step, we are going to specify the middle factors $R(t)$, which appeared in (39) and (43). The following proposition shows the existence of such factors, where the construction in the proof is completely explicit (although not unique). Moreover, although we noted that the factors $R(t)$ ought to be of a "simple" form, it turns out that the actually important point is that they are related by means of the equation $D(t) = R(t)W\widetilde{R}^{-1}(t)$ (in case of a factorization (39)) or the equation $D(t) = \widetilde{R}^{-1}(t)WR(t)$ (in case of a factorization (43)) to a factor $D(t)$ of the form (55).

Proposition 3.5 *Let $W \in \mathbb{C}^{N\times N}$ with $W^2 = I$, and assume that $D(t)$ is given by (55) such that $D(1) \sim D(-1) \sim W$. Then there exists a matrix function $R \in G\mathcal{B}^{N\times N}$ such that $D(t) = R(t)W\widetilde{R}^{-1}(t)$.*

Proof. We can assume that

$$W = T \operatorname{diag}(I_{\sigma_+}, -I_{\sigma_-})\, T^{-1}, \tag{73}$$

where σ_+ and σ_- are nonnegative integers with $\sigma_+ + \sigma_- = N$ and $T \in G\mathbb{C}^{N\times N}$. With the numbers $\alpha, \beta, \gamma, \delta$ defined as above in terms of the characteristic pairs appearing in $D(t)$, it follows from Proposition 3.4 (with $F(t) = D(t)$) that

$$\sigma_+ \;=\; \alpha+\beta \;=\; \alpha+\gamma \quad \text{ and } \quad \sigma_- \;=\; \beta+\delta \;=\; \gamma+\delta. \tag{74}$$

In particular, $\beta = \gamma$. Hence there exists a permutation matrix Π_1 such that

$$D(t) \;=\; \Pi_1 \operatorname{diag}\left(D_1(t), D_2(t), D_3(t)\right) \Pi_1^{-1}, \tag{75}$$

where

$$D_1(t) \;=\; \operatorname*{diag}_{0\le k\le\alpha}\, (t^{\varkappa_k^{(1)}}) \qquad \text{with } \varkappa_k^{(1)} \in \mathbb{Z} \text{ even}, \tag{76}$$

$$D_2(t) \;=\; \operatorname*{diag}_{0\le k\le\delta}\, (-t^{\varkappa_k^{(2)}}) \qquad \text{with } \varkappa_k^{(2)} \in \mathbb{Z} \text{ even}, \tag{77}$$

$$D_3(t) \;=\; \operatorname*{diag}_{0\le k\le\beta} \left(\begin{pmatrix} t^{\varkappa_k^{(3)}} & 0 \\ 0 & -t^{\varkappa_k^{(4)}} \end{pmatrix} \right) \qquad \text{with } \varkappa_k^{(3)}, \varkappa_k^{(4)} \in \mathbb{Z} \text{ odd}. \tag{78}$$

Moreover, there exists another permutation matrix Π_2 such that

$$W \;=\; T\Pi_2 \operatorname{diag}\left(I_\alpha, -I_\delta, X_\beta\right) \Pi_2^{-1} T^{-1}, \tag{79}$$

$$X_\beta \;=\; \operatorname*{diag}_{1\le k\le\beta} \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \tag{80}$$

We define

$$R_1(t) \;=\; \operatorname*{diag}_{0\le k\le\alpha}\, (t^{\varkappa_k^{(1)}/2}), \qquad\qquad R_2(t) \;=\; \operatorname*{diag}_{0\le k\le\delta}\, (t^{\varkappa_k^{(2)}/2}), \tag{81}$$

$$R_3(t) \;=\; \operatorname*{diag}_{0\le k\le\beta} \left(\frac{1}{2} \begin{pmatrix} t^{(\varkappa_k^{(3)}-\varkappa_k^{(4)})/2} + t^{(\varkappa_k^{(3)}+\varkappa_k^{(4)})/2} & t^{(\varkappa_k^{(3)}-\varkappa_k^{(4)})/2} - t^{(\varkappa_k^{(3)}+\varkappa_k^{(4)})/2} \\ 1 - t^{\varkappa_k^{(4)}} & 1 + t^{\varkappa_k^{(4)}} \end{pmatrix} \right). \tag{82}$$

It can be verified straightforwardly that

$$D_1(t) \;=\; R_1(t)\widetilde{R}_1^{-1}(t), \qquad D_2(t) \;=\; -R_2(t)\widetilde{R}_2^{-1}(t), \qquad D_3(t) \;=\; R_3(t)X_\beta\widetilde{R}_3^{-1}(t). \tag{83}$$

Hence, on defining R by

$$R(t) \;=\; \Pi_1 \operatorname{diag}\left(R_1(t), R_2(t), R_3(t)\right) \Pi_2^{-1} T^{-1}, \tag{84}$$

it follows that $D(t) = R(t)W\widetilde{R}^{-1}(t)$. $\square$

It is obvious that the previous proposition remains true if one replaces the expression $D(t) = R(t)W\widetilde{R}^{-1}(t)$ with $D(t) = \widetilde{R}^{-1}(t)WR(t)$. In fact, one just has to replace $R(t)$ with $\widetilde{R}^{-1}(t)$, which is possible due to the assumption (c) on the Banach algebra $\mathcal{B}$.

Besides $\mathcal{B}_+^{N\times N}$ and $\mathcal{B}_-^{N\times N}$, we need another subalgebra of $\mathcal{B}^{N\times N}$. Given $W \in \mathbb{C}^{N\times N}$ with $W^2 = I$, we define

$$\mathcal{B}_W^{N\times N} \quad = \quad \mathcal{B}^{N\times N} \cap (L^\infty)_W^{N\times N}. \tag{85}$$

It is easy to see that $\mathcal{B}_W^{N\times N}$ is an inverse closed Banach subalgebra of $\mathcal{B}^{N\times N}$.

The following theorem establishes the existence of two kinds of "asymmetric" factorizations within the Banach algebra $\mathcal{B}$ for a given function $A \in G\mathcal{B}^{N\times N}$.

Theorem 3.6 *Let $W \in \mathbb{C}^{N\times N}$ with $W^2 = I$, and assume that $A \in G\mathcal{B}^{N\times N}$. Then*

(a) there exists a factorization of $A(t)$ in the form

$$A(t) \quad = \quad A_-(t)R(t)A_0(t), \tag{86}$$

where $A_- \in G\mathcal{B}_-^{N\times N}$, $A_0 \in G\mathcal{B}_W^{N\times N}$, and $R \in G\mathcal{B}^{N\times N}$ such that $D(t) = R(t)W\widetilde{R}^{-1}(t)$ is of the form (55). Moreover,

$$F(t) \quad = \quad A_-(t)D(t)\widetilde{A}_-^{-1}(t) \tag{87}$$

represents an antisymmetric factorization of the function $F(t) = A(t)W\widetilde{A}^{-1}(t)$.

(b) there exists a factorization of $A(t)$ in the form

$$A(t) \quad = \quad A_0(t)R(t)A_+(t), \tag{88}$$

where $A_0 \in G\mathcal{B}_W^{N\times N}$, $A_- \in G\mathcal{B}_-^{N\times N}$, and $R \in G\mathcal{B}$ such that $D(t) = \widetilde{R}^{-1}(t)WR(t)$ is of the form (55). Moreover,

$$G(t) \quad = \quad \widetilde{A}_+^{-1}(t)D(t)A_+(t) \tag{89}$$

represents an antisymmetric factorization of the function $G(t) = \widetilde{A}^{-1}(t)WA(t)$.

Proof. From the definition of F it follows that $F \in G\mathcal{B}^{N\times N}$ and $F(t) = \widetilde{F}^{-1}(t)$. By Theorem 3.2 there exists an antisymmetric factorizations (87) with $A_- \in G\mathcal{B}_-^{N\times N}$ and $D(t)$ of the form (55).

From the definition of F it follows furthermore that $F(1) \sim F(-1) \sim W$, which in turn implies $D(1) \sim D(-1) \sim W$. Using Proposition 3.5 we obtain the existence of a function $R \in G\mathcal{B}^{N\times N}$ for which $D(t) = R(t)W\widetilde{R}^{-1}(t)$. Now we define

$$A_0(t) \quad = \quad R^{-1}(t)A_-^{-1}(t)A(t),$$

which implies immediately the validity of equation (86). Moreover,

$$W\widetilde{A}_0(t)W \;=\; W\widetilde{R}^{-1}(t)\widetilde{A}_-^{-1}(t)\widetilde{A}(t)W.$$

From

$$A(t)W\widetilde{A}^{-1}(t) \;=\; F(t) \;=\; A_-(t)D(t)\widetilde{A}_-^{-1}(t) \;=\; A_-(t)R(t)W\widetilde{R}^{-1}\widetilde{A}_-^{-1}(t)$$

we obtain

$$W\widetilde{A}_0(t)W \;=\; R^{-1}(t)A_-^{-1}(t)A(t) \;=\; A_0(t).$$

Hence $A_0 \in \mathcal{B}_W^{N\times N}$. Because, obviously, $A_0 \in G\mathcal{B}^{N\times N}$, we even have $A_0 \in G\mathcal{B}_W^{N\times N}$. This settles part (b).

The proof of part (b) is similar. By Theorem 3.2 (see also the remark made afterwards) and the facts that $G \in G\mathcal{B}^{N\times N}$ and $G(t) = \widetilde{G}^{-1}(t)$, there exists an antisymmetric factorization (89). From $G(1) \sim G(-1) \sim W$ we obtain $D(1) \sim D(-1) \sim W$. We apply again Proposition 3.5, but now with R replaced by $\widetilde{R}^{-1}$, in order to conclude the existence of a function $R \in G\mathcal{B}^{N\times N}$ for which $D(t) = \widetilde{R}^{-1}(t)WR(t)$. Finally, we define

$$A_0(t) \;=\; A(t)A_+^{-1}(t)R^{-1}(t),$$

apply the equation

$$\widetilde{A}^{-1}(t)WA(t) \;=\; G(t) \;=\; \widetilde{A}_+^{-1}(t)D(t)A_+(t) \;=\; \widetilde{A}_+^{-1}(t)\widetilde{R}^{-1}(t)WR(t)A_+(t)$$

and obtain in this way that $W\widetilde{A}_0(t)W = A_0(t)$. Hence $A_0 \in G\mathcal{B}_W^{N\times N}$. $\square$

The proof of the previous theorem reveals that the asymmetric factorization (86) of $A(t)$ can be constructed in an explicit way from the antisymmetric factorization of $F(t)$. Moreover, to each possible middle factor $D(t)$ (hence, to each possible collection of characteristic pairs), one can assign a corresponding function $R(t)$ which may appear as the middle factor in the asymmetric factorization. The fact that this assignment may be carried out in different ways does not affect the following considerations.

Similar statements hold, of course, also for the asymmetric factorization (88) of $A(t)$, which is connected with the antisymmetric factorization of $G(t)$.

4 Further properties of some classes of Toeplitz + Hankel operators

In this section we continue and conclude the study of the Toeplitz + Hankel operators $\mathcal{M}_W(A)$ and $\mathcal{N}_W(A)$ with $A \in \mathcal{B}^{N\times N}$. Notice first that it follows from Proposition 2.1 and the inverse closedness of $\mathcal{B}$ in $C(\mathbb{T})$ that these operators are Fredholm if and only if $A \in G\mathcal{B}^{N\times N}$.

In the case $A \in G\mathcal{B}^{N\times N}$ we will determine the dimension of the kernel and cokernel of $\mathcal{M}_W(A)$ and $\mathcal{N}_W(A)$ in terms of the characteristic pairs of an antisymmetric factorization of a certain associated function. Formulas for the inverses (if they exist) will also be presented.

For the following presentations, it is useful to introduce a function $\Theta : \{-1, 1\} \times \mathbb{Z} \to \mathbb{Z}$ which is defined by

$$\Theta(\varrho, \varkappa) \;=\; \left\{ \begin{array}{rl} \varkappa/2 & \text{if } \varkappa \text{ is even} \\ (\varkappa - \varrho)/2 & \text{if } \varkappa \text{ is odd.} \end{array} \right. \tag{90}$$

For the interpretation of the following results, it is also helpful to recall the notion of a pseudoinverse. Let A be a linear bounded operator acting on a Banach space X. A linear bounded operator $A^\dagger$ acting also on X is called a *pseudoinverse* of A if the relations

$$AA^\dagger A \;=\; A \quad \text{and} \quad A^\dagger AA^\dagger \;=\; A^\dagger \tag{91}$$

hold. One can show that a pseudoinverse of A exists if and only if the image of A is a complemented subspace in X, i.e., there exist a closed subspace X_0 of X such that $X = \operatorname{im} A \oplus X_0$. Hence each Fredholm operator possesses a pseudoinverse. Pseudoinverses are in general in not unique. However, if A is invertible, then $A^\dagger$ is uniquely determined and coincides with A^{-1}.

Lemma 4.1 *Let $D(t)$ be a matrix function of the form (55) with the characteristic pairs (57). Then $H(D)^* = H(D)$ and*

$$\dim\ker(I + H(D)) \;=\; \sum_{\varkappa_k>0} \Theta(\varrho_k, \varkappa_k). \tag{92}$$

Proof. First observe that $D^*(t) = \widetilde{D}(t)$. Consequently,

$$H(D)^* \;=\; (PM(D)JP)^* \;=\; PJM(D^*)P \;=\; PM(\widetilde{D}^*)JP \;=\; H(D).$$

Moreover, because $H(D) = \operatorname{diag}(H(\varrho_1 t^{\varkappa_1}), H(\varrho_2 t^{\varkappa_2}), \dots, H(\varrho_N t^{\varkappa_N}))$ is a diagonal operator it suffices to determine $\dim\ker(I + H(\varrho_k t^{\varkappa_k}))$ and to take the sum. If $\varkappa_k \le 0$, then $H(\varrho_k t^{\varkappa_k}) = 0$. Hence the corresponding dimension is zero. If $\varkappa_k > 0$, then the matrix representation of $H(\varrho_k t^{\varkappa_k})$ has entries ϱ_k only on the $\varkappa_k$-th diagonal, which connects the entries $(1, \varkappa_k)$ and $(\varkappa_k, 1)$, and has zero entries elsewhere. From this it is easy to see that the dimension equals $\Theta(\varrho_k, \varkappa_k)$. $\square$

Lemma 4.2 *Assume that $D(t)$ is of the form (55). Then*

$$T(\widetilde{D})H(D) \;=\; H(D)T(D) \;=\; 0 \quad \textit{and} \quad H(D)^3 \;=\; H(D). \tag{93}$$

Moreover, if we introduce

$$B \;=\; I + H(D), \qquad B^\dagger \;=\; I - H(D)^2 + \tfrac{1}{4}(H(D)^2 + H(D)), \tag{94}$$

then $B^\dagger BB^\dagger = B^\dagger$ and $BB^\dagger B = B$.

Proof. By considering the scalar case, $D(t) = \varrho_k t^{\varkappa_k}$ and distinguishing $\varkappa_k > \leq 0$ and $\varkappa_k > 0$, it can be seen straightforwardly that $T(\widetilde{D})H(D) = H(D)T(D) = 0$. Moreover, using (17) and the fact that $D(t) = \widetilde{D}^{-1}(t)$ it follows that

$$H(D)^3 \;=\; H(D)H(D)H(\widetilde{D}^{-1}) \;=\; H(D)(I - T(D)T(D^{-1})) \;=\; H(D).$$

In order to prove that $B^\dagger B B^\dagger = B^\dagger$ and $BB^\dagger B = B$, we introduce $p = I - H(D)^2$ and $q = (H(D) + H(D)^2)/2$. By just using the identity $H(D)^3 = H(D)$, one can verify that $p^2 = p$, $q^2 = q$, $pq = qp = 0$. Because $B = p + 2q$ and $B^\dagger = p + q/2$, the desired relations follow immediately. $\square$

Lemma 4.3 *Let $W \in \mathbb{C}^{N\times N}$ with $W^2 = I$, and assume that $R \in G\mathcal{B}^{N\times N}$ is given such that $D(t) = R(t)W\widetilde{R}^{-1}(t)$ is of the form (55). Introduce the operators B and $B^\dagger$ by (94), and*

$$A_1 \;=\; \mathcal{M}_W(R), \qquad\qquad A_2 \;=\; \mathcal{N}_W(R^{-1}), \tag{95}$$

$$A_1^\dagger \;=\; A_2 B^\dagger, \qquad\qquad A_2^\dagger \;=\; B^\dagger A_1. \tag{96}$$

Then $B = A_1 A_2$, $A_1^\dagger = \mathcal{N}_W(R^{-1})(I - \frac{1}{2}H(D))$, $A_2^\dagger = (I - \frac{1}{2}H(D))\mathcal{M}_W(R)$, and

$$A_1^\dagger A_1 A_1^\dagger \;=\; A_1^\dagger, \quad A_1 A_1^\dagger A_1 \;=\; A_1, \quad A_2^\dagger A_2 A_2^\dagger \;=\; A_2^\dagger, \quad A_2 A_2^\dagger A_2 \;=\; A_2. \tag{97}$$

Proof. Using formula (38), it follows that

$$\mathcal{M}_W(R)\mathcal{N}_W(R^{-1}) \;=\; T(RR^{-1}) + H(RW\widetilde{R}^{-1}) \;=\; I + H(D).$$

Hence $A_1 A_2 = B$. By using this, the relations of $A_1^\dagger = A_2 B^\dagger$ and $A_2^\dagger = B^\dagger A_1$, and formula $B^\dagger B B^\dagger = B^\dagger$ from Lemma 4.2, we obtain

$$A_1^\dagger A_1 A_1^\dagger \;=\; A_2 B^\dagger A_1 A_2 B^\dagger \;=\; A_2 B^\dagger B B^\dagger \;=\; A_2 B^\dagger \;=\; A_1^\dagger,$$

$$A_2^\dagger A_2 A_2^\dagger \;=\; B^\dagger A_1 A_2 B^\dagger A_1 \;=\; B^\dagger B B^\dagger A_1 \;=\; B^\dagger A_1 \;=\; A_2^\dagger.$$

Next, as an auxiliary step, we are going to establish the identities

$$H(D)A_1 \;=\; H(D)^2 A_1 \quad\text{ and }\quad A_2 H(D) \;=\; A_2 H(D)^2. \tag{98}$$

Indeed, using that $R = D\widetilde{R}W$ and formulas (17) and (18), it follows that

$$\begin{aligned} A_1 &= T(D\widetilde{R}W) + H(D\widetilde{R}) \\ &= T(D)T(\widetilde{R}W) + H(D)H(RW) + T(D)H(\widetilde{R}) + H(D)T(R) \\ &= H(D)A_1 + T(D)(T(\widetilde{R}W) + H(\widetilde{R})). \end{aligned}$$

Multiplying from the left with $H(D)$ and observing that $H(D)T(D) = 0$ by Lemma 4.2, we obtain the first identity in (98). Similarly, by using $R^{-1} = W\widetilde{R}^{-1}\widetilde{D}$, it follows that

$$\begin{aligned} A_2 &= T(W\widetilde{R}^{-1}\widetilde{D}) + H(R^{-1}D) \\ &= T(W\widetilde{R}^{-1})T(\widetilde{D}) + H(W\widetilde{R}^{-1})H(D) + T(R^{-1})H(D) + H(R^{-1})T(\widetilde{D}) \\ &= A_2H(D) + (T(W\widetilde{R}^{-1}) + H(R^{-1}))T(\widetilde{D}). \end{aligned}$$

Multiplying from the right with $H(D)$ by observing that $T(\widetilde{D})H(D) = 0$, we arrive at the second identity in (98).

Using this and the definition of $B^\dagger$, it follows that $A_1^\dagger = A_2(I - \frac{1}{2}H(D))$ and $A_2^\dagger = (I - \frac{1}{2}H(D))A_1$. Hence we obtain the desired expressions for $A_1^\dagger$ and $A_2^\dagger$.

Moreover, using the notation p and q introduced in the proof of Lemma 4.2, it is easy to see that $BB^\dagger = B^\dagger B = p + q$. Hence

$$BB^\dagger \;=\; B^\dagger B \;=\; I + \tfrac{1}{2}(H(D) - H(D)^2). \tag{99}$$

Combining this with (98) it follows that

$$BB^\dagger A_1 \;=\; A_1 \quad \text{and} \quad A_2B^\dagger B \;=\; A_2.$$

Now we are able to derive the remaining identities:

$$\begin{aligned} A_1A_1^\dagger A_1 \;&=\; A_1A_2B^\dagger A_1 \;=\; BB^\dagger A_1 \;=\; A_1, \\ A_2A_2^\dagger A_2 \;&=\; A_2B^\dagger A_1A_2 \;=\; A_2B^\dagger B \;=\; A_2. \end{aligned}$$

This completes the proof. □

Lemma 4.4 *Let A_1, A_2 and B as before. Then*

$$\dim\ker A_1^* \;=\; \dim\ker A_2 \;=\; \dim\ker B \tag{100}$$

Proof. Since $H(D)^* = H(D)$ by Lemma 4.1, it follows that $B^* = B$. The relation $B = A_1A_2$ stated in Lemma 4.3 implies that $\ker A_2 \subseteq \ker B$ and $\ker A_1^* \subseteq \ker B^*$. Moreover, because $A_2 = A_2A_2^\dagger A_2 = A_2B^\dagger A_1A_2 = A_1^\dagger B$, we obtain $\ker B \subseteq \ker A_2$. Similarly, since $A_1 = A_1A_1^\dagger A_1 = A_1A_2B^\dagger A_1 = BA_2^\dagger$, we arrive at $\ker B^* \subseteq \ker A_1^*$. □

Now we are able to establish formulas for the dimension of the kernel and cokernel of the operators $\mathcal{M}_W(R)$ and $\mathcal{N}_W(R)$, where $R(t)$ represent appropriate middle factors that are expected to appear in the asymmetric factorization. Notice the slightly modified notation in the following proposition, i.e., we are considering $\mathcal{N}_W(R^{-1})$ instead of $\mathcal{N}_W(R)$. The important point is, however, that $R(t)$ is related to a matrix function $D(t)$ of the form (55).

Proposition 4.5 *Let $W \in \mathbb{C}^{N\times N}$ with $W^2 = I$, and assume that $R \in G\mathcal{B}^{N\times N}$ is given such that $D(t) = R(t)W\widetilde{R}^{-1}(t)$ is of the form (55) with characteristic pairs (57). Then*

$$\dim\ker \mathcal{M}_W(R) \;=\; -\sum_{\varkappa_k<0}\Theta(\varrho_k,\varkappa_k), \qquad \dim\ker \mathcal{M}_W(R)^* \;=\; \sum_{\varkappa_k>0}\Theta(\varrho_k,\varkappa_k), \tag{101}$$

$$\dim\ker \mathcal{N}_W(R^{-1}) \;=\; \sum_{\varkappa_k>0}\Theta(\varrho_k,\varkappa_k), \qquad \dim\ker \mathcal{N}_W(R^{-1})^* \;=\; -\sum_{\varkappa_k<0}\Theta(\varrho_k,\varkappa_k). \tag{102}$$

Proof. The formulas for $\dim\ker \mathcal{M}_W(R)^*$ and $\dim\ker \mathcal{N}_W(R^{-1})$ follow immediately from Lemma 4.4 in connection with Lemma 4.1. Moreover, by the index formula stated in Proposition 2.1 it can be seen that

$$\begin{aligned}\operatorname{ind} \mathcal{M}_W(R) &= \operatorname{ind} T(R) \;=\; -\text{wind det}\, R,\\ \operatorname{ind} \mathcal{N}_W(R^{-1}) &= \operatorname{ind} T(R^{-1}) \;=\; -\text{wind det}\, R^{-1} \;=\; \text{wind det}\, R.\end{aligned}$$

Because $D(t) = R(t)W\widetilde{R}^{-1}(t)$, we have

$$\text{wind det}\, D \;=\; \text{wind det}\, R + \text{wind det}\, \widetilde{R}^{-1} \;=\; 2\,\text{wind det}\, R.$$

On the other hand,

$$\text{wind det}\, D \;=\; \sum_{k=1}^{N}\varkappa_k.$$

Combining all this, it follows that

$$\operatorname{ind} \mathcal{M}_W(R) \;=\; -\frac{1}{2}\sum_{k=1}^{N}\varkappa_k, \qquad \operatorname{ind} \mathcal{N}_W(R^{-1}) \;=\; \frac{1}{2}\sum_{k=1}^{N}\varkappa_k. \tag{103}$$

Since $D(1) \sim D(-1) \sim W$, we obtain from Proposition 3.4, in particular, that $\beta = \gamma$. We conclude from the definition of the function Θ that

$$\begin{aligned}\sum_{k=1}^{N}\Theta(\varrho_k,\varkappa_k) &= \frac{1}{2}\sum_{k=1}^{N}\varkappa_k - \frac{1}{2}\sum_{\varkappa_k \text{ odd}}\varrho_k\\ &= \frac{1}{2}\sum_{k=1}^{N}\varkappa_k + \frac{\gamma-\beta}{2} \;=\; \frac{1}{2}\sum_{k=1}^{N}\varkappa_k.\end{aligned} \tag{104}$$

Using the formulas

$$\begin{aligned}\operatorname{ind} \mathcal{M}_W(R) &= \dim\ker \mathcal{M}_W(R) - \dim\ker \mathcal{M}_W(R)^*,\\ \operatorname{ind} \mathcal{N}_W(R^{-1}) &= \dim\ker \mathcal{N}_W(R^{-1}) - \dim\ker \mathcal{N}_W(R^{-1})^*,\end{aligned}$$

it is easy to derive the remaining two formulas. $\square$

The following result determines the dimensions of the kernel and cokernel of the operators $\mathcal{M}_W(A)$ and $\mathcal{N}_W(A)$ for $A \in G\mathcal{B}^{N\times N}$ in terms of the characteristic pairs of an associated antisymmetric factorization problem. Note that the existence of this antisymmetric factorization is ensured by Theorem 3.6.

Moreover, we give expressions for the pseudoinverses of the above operator, which are the inverses in case of invertibility. It should also be observed that in the formulation of the following theorem we need not make reference to the asymmetric factorizations, although they are, of course, used in the proof.

Theorem 4.6 *Let $W \in \mathbb{C}^{N\times N}$ with $W^2 = I$, and let $A \in G\mathcal{B}^{N\times N}$.*

(a) Assume that an antisymmetric factorization of the function $F(t) = A(t)W\widetilde{A}^{-1}(t)$ is given by $F(t) = A_-(t)D(t)\widetilde{A}_-^{-1}(t)$, where $A_- \in G\mathcal{B}_-^{N\times N}$ and $D(t)$ is of the form (55) with the characteristic pairs (57). Then

$$\dim\ker \mathcal{M}_W(A) \;=\; -\sum_{\varkappa_k<0} \Theta(\varrho_k, \varkappa_k), \tag{105}$$

$$\dim\ker \mathcal{M}_W(A)^* \;=\; \sum_{\varkappa_k>0} \Theta(\varrho_k, \varkappa_k). \tag{106}$$

Moreover, a pseudoinverse of $\mathcal{M}_W(A)$ is given by

$$\mathcal{N}_W(A^{-1}A_-)(I - \tfrac{1}{2}H(D))T(A_-^{-1}). \tag{107}$$

(b) Assume that an antisymmetric factorization of the function $G(t) = \widetilde{A}^{-1}(t)WA(t)$ is given by $G(t) = \widetilde{A}_+^{-1}(t)D(t)A_+(t)$, where $A_+ \in G\mathcal{B}_+^{N\times N}$ and $D(t)$ is of the form (55) with the characteristic pairs (57). Then

$$\dim\ker \mathcal{N}_W(R) \;=\; -\sum_{\varkappa_k<0} \Theta(-\varrho_k, \varkappa_k), \tag{108}$$

$$\dim\ker \mathcal{N}_W(R)^* \;=\; \sum_{\varkappa_k>0} \Theta(-\varrho_k, \varkappa_k). \tag{109}$$

Moreover, a pseudoinverse of $\mathcal{N}_W(A)$ is given by

$$T(A_+^{-1})(I - \tfrac{1}{2}H(D^{-1}))\mathcal{M}_W(A_+A^{-1}). \tag{110}$$

Proof. Let us first consider case (a). By Theorem 3.6 we can assume that we are given an asymmetric factorization $A(t) = A_-(t)R(t)A_0(t)$ with the conditions on the factors stated

there. In addition, we are given an antisymmetric factorization $F(t) = A_-(t)D(t)\widetilde{A}_-^{-1}(t)$ with $D(t) = R(t)W\widetilde{R}^{-1}(t)$ of the function $F(t) = A(t)W\widetilde{A}^{-1}(t)$. From (32) and (34) it follows that

$$\mathcal{M}_W(A) \;=\; \mathcal{M}_W(A_-)\mathcal{M}_W(R)\mathcal{M}_W(A_0),$$

where both $\mathcal{M}_W(A_-) = T(A_-)$ and $\mathcal{M}_W(A_0)$ are invertible. There inverses are equal to $T(A_-^{-1})$ and $\mathcal{M}_W(A_0^{-1})$, respectively. Hence the dimension of the kernel and cokernel of $\mathcal{M}_W(A)$ is equal to that of $\mathcal{M}_W(R)$, which, in turn, has been given in Proposition 4.5.

Next we need to take into account the following fact, which can be proved straightforwardly: if an operator S_1 has a pseudoinverse $S_1^\dagger$ and $S_2 = US_2V$ where U and V are invertible operators, then a pseudoinverse of S_2 is given by $S_2^\dagger = V^{-1}S_1^\dagger U^{-1}$.

It follows from Lemma 4.3 that a pseudoinverse of $\mathcal{M}_W(R) = A_1$ is given by $\mathcal{N}_W(R^{-1})(I-\frac{1}{2}H(D)) = A_1^\dagger$. Consequently, a pseudoinverse of $\mathcal{M}_W(A)$ is given by

$$\mathcal{M}_W(A_0^{-1})\mathcal{N}_W(R^{-1})(I - \tfrac{1}{2}H(D))T(A_-^{-1}).$$

Now we use formula (36) and (33) in order to conclude that

$$\mathcal{M}_W(A_0^{-1})\mathcal{N}_W(R^{-1}) \;=\; \mathcal{N}_W(A_0^{-1})\mathcal{N}_W(R^{-1}) \;=\; \mathcal{N}_W(A_0^{-1}R^{-1}).$$

Remark that $A_0^{-1}R^{-1} = A^{-1}A_-$ (because this is just the equation $A = A_-RA_0$). Combining these last facts, we arrive at the desired expression for the pseudoinverse.

Case (b) can be treated in the same way, but we give the complete proof because the notation differs sometimes here in comparison with previous results. First of all, we may assume that we are given an asymmetric factorization $A(t) = A_0(t)R(t)A_+(t)$ as has been stated in Theorem 3.6. Moreover, we are given an antisymmetric factorization $G(t) = \widetilde{A}_+^{-1}(t)D(t)A_+(t)$ with $D(t) = \widetilde{R}^{-1}(t)WR(t)$ of the function $G(t) = \widetilde{A}^{-1}(t)WA(t)$. From (33) and (35) it follows that

$$\mathcal{N}_W(A) \;=\; \mathcal{N}_W(A_0)\mathcal{N}_W(R)\mathcal{N}_W(A_+),$$

where $\mathcal{N}_W(A_0)$ and $\mathcal{N}_W(A_+) = T(A_+)$ are invertible. The inverses are $\mathcal{N}_W(A_0^{-1})$ and $T(A_+^{-1})$, respectively. The above relation $D(t) = \widetilde{R}^{-1}(t)WR(t)$ can be rewritten as $D^{-1}(t) = R^{-1}(t)W\widetilde{R}(t)$. We now have to apply Proposition 4.5 and Lemma 4.3 with $R(t)$ replaced with $R^{-1}(t)$ and $D(t)$ replaced with $D^{-1}(t)$. Correspondingly, the characteristic pairs $(\varrho_k, \varkappa_k)$ have to be replaced with $(\varrho_k, -\varkappa_k)$. We arrive at the formulas

$$\dim\ker\mathcal{N}_W(R) \;=\; \sum_{\varkappa_k<0}\Theta(\varrho_k, -\varkappa_k), \qquad \dim\ker\mathcal{N}_W(R)^* \;=\; -\sum_{\varkappa_k>0}\Theta(\varrho_k, -\varkappa_k).$$

It remains to note that $\Theta(\varrho_k, -\varkappa_k) = -\Theta(-\varrho_k, \varkappa_k)$ in order to conclude the desired formulas for the dimension of the kernel and cokernel of $\mathcal{N}_W(A)$.

Also in regard to the pseudoinverse we have to apply Lemma 4.3, but with $R(t)$ replaced with $R^{-1}(t)$ and $D(t)$ replaced with $D^{-1}(t)$. It follows that the pseudoinverse of $\mathcal{N}_W(R) = A_2$ is given by $(I - \frac{1}{2}H(D^{-1}))\mathcal{M}_W(R^{-1}) = A_2^{\dagger}$. As before, we obtain that a pseudoinverse of $\mathcal{N}_W(A)$ is given by

$$T(A_+^{-1})(I - \tfrac{1}{2}H(D^{-1}))\mathcal{M}_W(R^{-1})\mathcal{N}_W(A_0^{-1}).$$

Using formulas (36) and (32) we derive

$$\mathcal{M}_W(R^{-1})\mathcal{N}_W(A_0^{-1}) \;=\; \mathcal{M}_W(R^{-1})\mathcal{M}_W(A_0^{-1}) \;=\; \mathcal{M}_W(R^{-1}A_0^{-1}).$$

The desired pseudoinverse of $\mathcal{M}_W$ is now obtained by piecing together these last facts in connection with $R^{-1}A_0^{-1} = A_+A^{-1}$, which is just the factorization $A = A_0RA_+$ rewritten. □

At the end of this section we consider some simple consequences of the previous theorem. In particular, we state the necessary and sufficient conditions for the invertibility of the operators $\mathcal{M}_W(A)$ and $\mathcal{N}_W(A)$.

Corollary 4.7 *Let $W \in \mathbb{C}^{N\times N}$ with $W^2 = I$, and assume $A \in G\mathcal{B}^{N\times N}$.*

(a) The operator $\mathcal{M}_W(A)$ is invertible if and only if the function $F(t) = A(t)W\widetilde{A}^{-1}(t)$ admits an antisymmetric factorization with characteristic pairs $(\varrho_k, \varkappa_k)$ which are all contained in the set

$$\Big\{\, (-1,-1),\, (-1,0),\, (1,0),\, (1,1) \,\Big\}. \tag{111}$$

(b) The operator $\mathcal{N}_W(A)$ is invertible if and only if the function $G(t) = \widetilde{A}^{-1}(t)WA(t)$ admits an antisymmetric factorization with characteristic pairs $(\varrho_k, \varkappa_k)$ which are all contained in the set

$$\Big\{\, (1,-1),\, (-1,0),\, (1,0),\, (-1,1) \,\Big\}. \tag{112}$$

Proof. The operators are invertible if and only if the sums in (105) and (106), or, (108) and (109), respectively are zero. Notice that the different terms appearing there are all nonnegative integers. Hence they must be equal to zero. It remains to remark that $\Theta(\varrho,\varkappa) = 0$ if and only if $\varkappa = 0$ or $\varkappa = \varrho = 1$ or $\varkappa = \varrho = -1$. □

The previous result takes a much simpler form in the two special cases where $W = I$ or $W = -I$. In fact, we can apply Proposition 3.4 and recall the definition of the numbers $\alpha, \beta, \gamma, \delta$.

In the case where $W = I$, we have $F(1) = F(-1) = I$ and $G(1) = G(-1) = I$. Hence Proposition 3.4 implies that $\alpha = N$ and $\beta = \gamma = \delta = 0$. Hence among the pairs given in (111) or (112) only the pair $(1,0)$ can occur. The result is that the operator $\mathcal{M}_I(A)$ ($\mathcal{N}_I(A)$, resp.) is invertible if and only if the function $F(t) = A(t)\widetilde{A}^{-1}(t)$ ($G(t) = \widetilde{A}^{-1}(t)A(t)$, resp.) admits an antisymmetric factorization with all characteristic pairs equal to $(1,0)$.

In the case where $W = -I$, we obtain in a similar way the result that the operator $\mathcal{M}_{-I}(A)$ ($\mathcal{N}_{-I}(A)$, resp.) is invertible if and only if the function $F(t) = -A(t)\widetilde{A}^{-1}(t)$ ($G(t) = -\widetilde{A}^{-1}(t)A(t)$, resp.) admits an antisymmetric factorization with all characteristic pairs equal to $(-1, 0)$.

5 Singular integral operators with flip

In this section, we study the properties of singular integral operators with flip. In particular, we obtain results for the dimension of the kernel and cokernel in the case of Fredholmness under the assumption that the generating functions belongs to the Banach algebra $\mathcal{B}^{N\times N}$.

In what follows, when we are given the matrix functions $a, b, c, d \in L^\infty(\mathbb{T})^{N\times N}$, we associate a matrix function of twice the matrix size,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L^\infty(\mathbb{T})^{2N\times 2N}. \tag{113}$$

Moreover, we let W stand for the following constant matrix of size $2N \times 2N$,

$$W = \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}. \tag{114}$$

Finally, to a matrix function $A \in L^\infty(\mathbb{T})^{2N\times 2N}$ given as above, we associate a matrix function $\widehat{A} \in L^\infty(\mathbb{T})^{2N\times 2N}$ defined by

$$\widehat{A}(t) = W\widetilde{A}(t)W. \tag{115}$$

Next we introduce, for given $A \in L^\infty(\mathbb{T})^{2N\times 2N}$, the following operators

$$\mathcal{T}(A) = \Big(P,\, JP\Big) M(A) \begin{pmatrix} P \\ PJ \end{pmatrix}, \tag{116}$$

$$\mathcal{H}(A) = \Big(P,\, JP\Big) M(A) \begin{pmatrix} Q \\ QJ \end{pmatrix}. \tag{117}$$

Using the basic relations for the operators P, Q, J, and $M(A)$, it is easy to see that then

$$\mathcal{T}(\widehat{A}) = \Big(Q,\, JQ\Big) M(A) \begin{pmatrix} Q \\ QJ \end{pmatrix}, \tag{118}$$

$$\mathcal{H}(\widehat{A}) = \Big(Q,\, JQ\Big) M(A) \begin{pmatrix} P \\ PJ \end{pmatrix}. \tag{119}$$

Moreover, given $A, B \in L^\infty(\mathbb{T})^{2N\times 2N}$, the following relations hold:

$$\mathcal{T}(AB) = \mathcal{T}(A)\mathcal{T}(B) + \mathcal{H}(A)\mathcal{H}(\widehat{B}), \tag{120}$$

$$\mathcal{H}(AB) = \mathcal{T}(A)\mathcal{H}(B) + \mathcal{H}(A)\mathcal{T}(\widehat{B}). \tag{121}$$

In fact, they are essentially a consequence of the identity

$$\begin{pmatrix} P \\ PJ \end{pmatrix} \Big(P,\, JP\Big) + \begin{pmatrix} Q \\ QJ \end{pmatrix} \Big(Q,\, JQ\Big) \;=\; \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

The formal resemblance to the formulas (17) and (18) is obvious.

Finally, we define the operators

$$\Phi(A) \;=\; \mathcal{T}(A) + \mathcal{H}(A) \;=\; \Big(P,\, JP\Big) M(A) \begin{pmatrix} I \\ J \end{pmatrix}, \tag{122}$$

$$\Psi(A) \;=\; \mathcal{T}(A) + \mathcal{H}(\widehat{A}) \;=\; \Big(I,\, J\Big) M(A) \begin{pmatrix} P \\ PJ \end{pmatrix}. \tag{123}$$

These operators are the *singular integral operators with flip* which we intend to study in this section. Indeed, if A is given by (113), then

$$\Phi(A) \;=\; PM(a) + PJM(\tilde{b}) + QJM(c) + QM(\tilde{d}), \tag{124}$$

$$\Psi(A) \;=\; M(a)P + M(b)JQ + M(\tilde{c})JP + M(\tilde{d})Q. \tag{125}$$

Using the formulas (120) and (121), formulas analogous to (29) and (30) can be derived:

$$\Phi(AB) \;=\; \Phi(A)\Phi(B) + \mathcal{H}(A)\Phi(\widehat{B} - B), \tag{126}$$

$$\Psi(AB) \;=\; \Psi(A)\Psi(B) + \Psi(\widehat{A} - A)\mathcal{H}(\widehat{B}). \tag{127}$$

Indeed,

$$\begin{aligned} \Phi(AB) &= \mathcal{T}(AB) + \mathcal{H}(AB) \\ &= \mathcal{T}(A)\mathcal{T}(B) + \mathcal{H}(A)\mathcal{H}(\widehat{B}) + \mathcal{T}(A)\mathcal{H}(B) + \mathcal{H}(A)\mathcal{T}(\widehat{B}) \\ &= \mathcal{T}(A)\Phi(B) + \mathcal{H}(A)\Phi(\widehat{B}) \\ &= \Phi(A)\Phi(B) + \mathcal{H}(A)\Phi(\widehat{B} - B). \end{aligned}$$

Moreover,

$$\begin{aligned} \Psi(AB) &= \mathcal{T}(AB) + \mathcal{H}(\widehat{A}\widehat{B}) \\ &= \mathcal{T}(A)\mathcal{T}(B) + \mathcal{H}(A)\mathcal{H}(\widehat{B}) + \mathcal{T}(\widehat{A})\mathcal{H}(\widehat{B}) + \mathcal{H}(\widehat{A})\mathcal{T}(B) \\ &= \Psi(A)\mathcal{T}(B) + \Psi(\widehat{A})\mathcal{H}(\widehat{B}) \\ &= \Psi(A)\Psi(B) + \Psi(\widehat{A} - A)\mathcal{H}(\widehat{B}). \end{aligned}$$

The corresponding "simplifications", where multiplicativity holds, read as follows:

$$\Phi(AB) \;=\; \Phi(A)\Phi(B) \qquad \text{if } A \in (\overline{H^\infty})^{2N\times 2N} \text{ or } B \in (L^\infty)_W^{2N\times 2N}; \tag{128}$$

$$\Psi(AB) \;=\; \Psi(A)\Psi(B) \qquad \text{if } A \in (L^\infty)_W^{2N\times 2N} \text{ or } B \in (H^\infty)^{2N\times 2N}. \tag{129}$$

Here W is given by (114). Notice that the Banach algebra $(L^\infty)_W^{2N\times 2N}$ is equal to

$$\Big\{ A \in L^\infty(\mathbb{T})^{2N\times 2N} \ : \ \widehat{A} = A \Big\}. \tag{130}$$

Also the counterpart to formula (38) can be established:

$$\Phi(A)\Psi(B) \ = \ \mathcal{T}(AB) + \mathcal{H}(A\widehat{B}). \tag{131}$$

Indeed, using (120) and (121) it follows that

$$\begin{aligned}
\Phi(A)\Psi(B) &= \Big(\mathcal{T}(A) + \mathcal{H}(A)\Big)\Big(\mathcal{T}(B) + \mathcal{H}(\widehat{B})\Big) \\
&= \mathcal{T}(A)\mathcal{T}(B) + \mathcal{H}(A)\mathcal{H}(\widehat{B}) + \mathcal{H}(A)\mathcal{T}(B) + \mathcal{T}(A)\mathcal{H}(\widehat{B}) \\
&= \mathcal{T}(AB) + \mathcal{H}(A\widehat{B}).
\end{aligned}$$

The analogy of these formulas in comparison with previous formulas finds its crystal explanation in the following result.

Proposition 5.1 *The mapping Ξ defined by*

$$\Xi \ : \ \mathcal{L}(L^2)^{N\times N} \to \mathcal{L}(H^2)^{2N\times 2N}, \ \ X \mapsto \begin{pmatrix} P \\ PJ \end{pmatrix} X \Big(P, \ JP\Big) \tag{132}$$

represents a C^*-algebra isomorphism between $\mathcal{L}(L^2)^{N\times N}$ and $\mathcal{L}(H^2)^{2N\times 2N}$. In particular, the mapping Ξ acts as follows:

$$\Xi : \mathcal{T}(A) \mapsto T(A), \qquad \Xi : \mathcal{H}(A) \mapsto H(AW), \tag{133}$$

$$\Xi : \Phi(A) \mapsto \mathcal{M}_W(A), \qquad \Xi : \Psi(A) \mapsto \mathcal{N}_W(A), \tag{134}$$

for $A \in L^\infty(\mathbb{T})^{2N\times 2N}$, where W is given by (114).

Proof. The first assertion follows from the fact that the linear operators

$$\Big(P, \ JP\Big) : (H^2)^{2N} \to (L^2)^N \quad \text{and} \quad \begin{pmatrix} P \\ PJ \end{pmatrix} : (L^2)^N \to (H^2)^{2N} \tag{135}$$

are Hilbert space isometries and are both the inverse and the adjoint of each other. In fact,

$$\begin{aligned}
\Big(P, \ JP\Big)\begin{pmatrix} P \\ PJ \end{pmatrix} &= I, \\
\begin{pmatrix} P \\ PJ \end{pmatrix}\Big(P, \ JP\Big) &= \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}.
\end{aligned}$$

In order to prove (133) and (134) it suffices to recall the definitions (116), (117), (122), (123), to use the last identity and the relation

$$\begin{pmatrix} Q \\ QJ \end{pmatrix} = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}\begin{pmatrix} P \\ PJ \end{pmatrix} = W\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}\begin{pmatrix} P \\ PJ \end{pmatrix}.$$

One has also to use the definition of $\mathcal{M}_W(A)$ and $\mathcal{N}_W(A)$ and the fact that $\widehat{A} = W\widetilde{A}W$. $\square$

The importance of the previous proposition is that it says that the above singular integral operators with flip are unitarily equivalent to Toeplitz + Hankel operators. These Toeplitz + Hankel operators fall exactly in the classes which were studied in the previous sections. Hence it is possible to reduce the study of the several properties of singular integral operators with flip to the corresponding problems for these Toeplitz + Hankel operators, which has already been done.

In regard to Proposition 2.1 and Proposition 2.2, the following result is an immediate consequence.

Proposition 5.2 *Let $A \in L^\infty(\mathbb{T})^{2N\times 2N}$.*

(a) If $\Phi(A)$ is Fredholm, then $A \in G(L^\infty(\mathbb{T})^{2N\times 2N})$.

(b) If $\Psi(A)$ is Fredholm, then $A \in G(L^\infty(\mathbb{T})^{2N\times 2N})$.

Now assume that $A \in C(\mathbb{T})^{2N\times 2N}$. Then

(c) $\Phi(A)$ is Fredholm if and only if $A \in G(C(\mathbb{T})^{2N\times 2N})$.

(d) $\Psi(A)$ is Fredholm if and only if $A \in G(C(\mathbb{T})^{2N\times 2N})$.

Moreover, if this is true, then $\operatorname{ind}\Phi(A) = \operatorname{ind}\Psi(A) = -\operatorname{wind}\det A$.

Another result concerns the case where $A \in (L^\infty)^{2N\times 2N}_W$ with W given by (114). As we will see shortly, this case is trivial. If A is given by (113) and $\widehat{A} = A$, then $d = \tilde{a}$ and $c = \tilde{b}$. In other words,

$$\Phi(A) = \Psi(A) = M(a) + M(b)J, \tag{136}$$

which is an operator composed of multiplication operators and the flip operator, but without the usual singular integral operator $S = P - Q$. Compare in this connection the first equality in this formula with the identity (36).

For completeness sake, we state the corresponding invertibility and Fredholm criteria for operators (136), which follow from Corollary 2.3 by means of Proposition 5.1. It can be proved also by different, more straightforward considerations.

Corollary 5.3 *Let $A \in (L^\infty)_W^{2N\times 2N}$, where W is given by (114). Then the following is equivalent:*

(i) $A \in G(L^\infty)_W^{2N\times 2N}$.

(ii) $\Phi(A) = \Psi(A)$ *is invertible.*

(iii) $\Phi(A) = \Psi(A)$ *is Fredholm.*

If this is fulfilled, then the inverse of $\Phi(A) = \Psi(A)$ is given by $\Phi(A^{-1}) = \Psi(A^{-1})$.

Now we turn to the case in which we are actually interestated in, namely the operators $\Phi(A)$ and $\Psi(A)$ with $A \in \mathcal{B}^{2N\times 2N}$. As before we assume that the Banach algebra $\mathcal{B}$ possesses the properties (a)–(d) stated at the beginning of Section 3.

Because the Banach algebra $\mathcal{B}$ is inverse closed in $C(\mathbb{T})$, Proposition 5.2(cd) implies that, for given $A \in \mathcal{B}^{2N\times 2N}$, the operator $\Phi(A)$ ($\Psi(A)$, resp.) is a Fredholm operator if and only if $A \in G\mathcal{B}^{2N\times 2N}$. Similar as in Section 4, and, of course, referring to these results, we will determine the dimension of the kernel and cokernel of $\Phi(A)$ and $\Psi(A)$ in terms of the characteristic pairs of an antisymmetric factorization of a certain associated function. Formulas for pseudoinverses (which are the inverses in the case of invertibility) will also be presented.

Theorem 5.4 *Let W be given by (114), and let $A \in G\mathcal{B}^{2N\times 2N}$.*

(a) Assume that an antisymmetric factorization of the function $F(t) = A(t)W\widetilde{A}^{-1}(t)$ is given by $F(t) = A_-(t)D(t)\widetilde{A}_-^{-1}(t)$, where $A_- \in G\mathcal{B}_-^{N\times N}$ and $D(t)$ is of the form (55) with the characteristic pairs (57). Then

$$\dim\ker\Phi(A) = -\sum_{\varkappa_k<0}\Theta(\varrho_k,\varkappa_k), \tag{137}$$

$$\dim\ker\Phi(A)^* = \sum_{\varkappa_k>0}\Theta(\varrho_k,\varkappa_k). \tag{138}$$

Moreover, a pseudoinverse of $\Phi(A)$ is given by

$$\Psi(A^{-1}A_-)(I - \tfrac{1}{2}\mathcal{H}(DW))\mathcal{T}(A_-^{-1}). \tag{139}$$

(b) Assume that an antisymmetric factorization of the function $G(t) = \widetilde{A}^{-1}(t)WA(t)$ is given by $G(t) = \widetilde{A}_+^{-1}(t)D(t)A_+(t)$, where $A_+ \in G\mathcal{B}_+^{N\times N}$ and $D(t)$ is of the form (55) with the characteristic pairs (57). Then

$$\dim\ker\Psi(A) = -\sum_{\varkappa_k<0}\Theta(-\varrho_k,\varkappa_k), \tag{140}$$

$$\dim\ker\Psi(A)^* = \sum_{\varkappa_k>0}\Theta(-\varrho_k,\varkappa_k). \tag{141}$$

Moreover, a pseudoinverse of $\Psi(A)$ is given by

$$\mathcal{T}(A_+^{-1})(I - \tfrac{1}{2}\mathcal{H}(D^{-1}W))\Phi(A_+A^{-1}) \tag{142}$$

Proof. The proof is based on Theorem 4.6 and Proposition 5.1. Because $\Phi(A)$ and $\Psi(A)$ are unitarily equivalent to $\mathcal{M}_W(A)$ and $\mathcal{N}_W(A)$, respectively, the formulas for the dimension of the kernel and cokernel follow immediately. In order to show that the above expression are indeed pseudoinverses, one can apply the C*-algebra isomorphism Ξ introduced in Proposition 5.1 to these operators. Using the formulas stated there, one obtains

$$\begin{array}{rcl} \Xi &:& \Psi(A^{-1}A_-)(I - \frac{1}{2}\mathcal{H}(DW))\mathcal{T}(A_-^{-1}) \quad \mapsto \quad \mathcal{N}_W(A^{-1}A_-)(I - \frac{1}{2}H(D))T(A_-^{-1}), \\ \Xi &:& \mathcal{T}(A_+^{-1})(I - \frac{1}{2}\mathcal{H}(D^{-1}W))\Phi(A_+A^{-1}) \quad \mapsto \quad T(A_+^{-1})(I - \frac{1}{2}H(D^{-1}))\mathcal{M}_W(A_+A^{-1}). \end{array}$$

The operators on the right hand side are exactly the expressions (107) and (110) for the pseudoinverses of $\mathcal{M}_W(A)$ and $\mathcal{N}_W(A)$. The observation that an operator $X^\dagger$ is a pseudoinverse of an operator X if and only if $\Xi(X^\dagger)$ is a pseudoinverse of $\Xi(X)$ completes the proof. □

The corresponding invertibility criteria reads as follows (compare Corollary 4.7).

Corollary 5.5 *Let W be given by (114), and assume $A \in G\mathcal{B}^{2N\times 2N}$.*

(a) The operator $\Phi(A)$ is invertible if and only if the function $F(t) = A(t)W\widetilde{A}^{-1}(t)$ admits an antisymmetric factorization with characteristic pairs $(\varrho_k, \varkappa_k)$ which are all contained in the set

$$\Big\{ \,(-1,-1),\,(-1,0),\,(1,0),\,(1,1)\, \Big\}. \tag{143}$$

(b) The operator $\Psi(A)$ is invertible if and only if the function $G(t) = \widetilde{A}^{-1}(t)WA(t)$ admits an antisymmetric factorization with characteristic pairs $(\varrho_k, \varkappa_k)$ which are all contained in the set

$$\Big\{ \,(1,-1),\,(-1,0),\,(1,0),\,(-1,1)\, \Big\}. \tag{144}$$

6 General Toeplitz + Hankel operators

In this section we study Toeplitz + Hankel operators $T(a)+H(b)$ where no “a priori” relation between a and b is assumed.

It has been stated in Proposition 2.1 that the Fredholmness of $T(a) + H(b)$ with $a, b \in L^\infty(\mathbb{T})^{N\times N}$ implies $a \in G(L^\infty(\mathbb{T})^{N\times N})$. Moreover, in the case where $a, b \in C(\mathbb{T})^{N\times N}$ the necessary and sufficient criteria for Fredholmness has been stated in Proposition 2.2.

We are going to consider the case where $a, b \in \mathcal{B}^{N\times N}$. It follows as before from the inverse closedness of $\mathcal{B}$ in $C(\mathbb{T})$ that $T(a) + H(b)$ with $a, b \in \mathcal{B}^{N\times N}$ is Fredholm if and only if $a \in G\mathcal{B}^{N\times N}$. The dimension of the kernel and cokernel in the case of Fredholmness reads as follows.

Theorem 6.1 *Let $a \in G\mathcal{B}^{N\times N}$, $b \in \mathcal{B}^{N\times N}$, and W be given by (114). Introduce the functions*

$$A(t) = \begin{pmatrix} a(t) & b(t) \\ 0 & I_N \end{pmatrix} \in \mathcal{B}^{2N\times 2N}, \tag{145}$$

$$F(t) = A(t)W\widetilde{A}^{-1}(t) = \begin{pmatrix} b(t)\tilde{a}^{-1}(t) & a(t) - b(t)\tilde{a}^{-1}(t)\tilde{b}(t) \\ \tilde{a}^{-1}(t) & -\tilde{a}^{-1}(t)\tilde{b}(t) \end{pmatrix} \in \mathcal{B}^{2N\times 2N}. \tag{146}$$

If the characteristic pairs of the antisymmetric factorization of $F(t) = A_-(t)D(t)\widetilde{A}_-^{-1}(t)$ are given by (57), then

$$\dim\ker(T(a) + H(b)) = -\sum_{\varkappa_k<0} \Theta(\varrho_k, \varkappa_k), \tag{147}$$

$$\dim\ker(T(a) + H(b))^* = \sum_{\varkappa_k>0} \Theta(\varrho_k, \varkappa_k). \tag{148}$$

Moreover, if we write

$$A^{-1}A_- = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad A_-^{-1} = \Big(v_1,\ v_2\Big), \tag{149}$$

with $u_1, u_2 \in \mathcal{B}^{N\times 2N}$ and $v_1, v_2 \in \mathcal{B}^{2N\times N}$, then a pseudoinverse of $T(a) + H(b)$ is given by

$$\Big(T(u_1) + H(\tilde{u}_2)\Big)\Big(P - \tfrac{1}{2}H(D)\Big)T(v_1) \tag{150}$$

Proof. The dimension the kernel and cokernel of the Toeplitz + Hankel operator $T(a)+H(b)$, which is defined on $(H^2)^N$, coincides with that of the operator

$$X = PM(a)P + PM(b)JP + Q,$$

which is defined on $(L^2)^N$. Now we write

$$PM(a)P + PM(b)JP + Q = (I - PM(a)Q - PM(b)JQ)(PM(a) + PM(b)J + Q)$$

Because $I - Y = PM(a)Q + PM(b)JQ$ is nilpotent, the first expression on the right hand side (i.e., the operator Y) is invertible. Hence we have to determine the dimension of the kernel and cokernel of

$$PM(a) + PM(b)J + Q,$$

which is just the singular integral operator $\Phi(A)$ with $A(t)$ given as above. Now the result follows from Theorem 5.4.

As to the pseudoinverse, we first remark that a pseudoinverse of $T(a)+H(b)$ is given by $PX^\dagger P$, where $X^\dagger$ is a pseudoinverse of the above operator X. Since $X = Y\Phi(A)$, it follows that $X^\dagger = (\Phi(A))^\dagger Y^{-1}$. Hence $PX^\dagger P = P(\Phi(A))^\dagger P$ because $P = Y^{-1}P$ as can easily be seen. From Theorem 5.4(a) we conclude that $(\Phi(A))^\dagger$ may be given by

$$\Psi(A^{-1}A_-)(I - \tfrac{1}{2}\mathcal{H}(DW))\mathcal{T}(A_-^{-1})$$

Using the definition of the operators occurring there, we obtain that this is equal to

$$\begin{aligned}
&\Big(I, J\Big) M(A^{-1}A_-) \begin{pmatrix} P \\ PJ \end{pmatrix} \Big(I - \frac{1}{2}\Big(P, JP\Big) M(D) \begin{pmatrix} QJ \\ Q \end{pmatrix}\Big) \Big(P, JP\Big) M(A_-^{-1}) \begin{pmatrix} P \\ PJ \end{pmatrix} \\
&= \quad \Big(I, J\Big) M(A^{-1}A_-) \Big(P - \frac{1}{2} PM(D)JP\Big) M(A_-^{-1}) \begin{pmatrix} P \\ PJ \end{pmatrix}.
\end{aligned}$$

Hence $PX^\dagger P = P(\Phi(A))^\dagger P$ equals

$$\Big(P, PJ\Big) M \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} P\Big(P - \frac{1}{2}H(D)\Big) PM(v_1, v_2) \begin{pmatrix} P \\ 0 \end{pmatrix},$$

which in turn is equal to the operator (150). □

We want to emphasize that the matrices u_k and v_k are of size $N \times 2N$ and $2N \times N$, respectively, whereas D is of size $2N \times 2N$. The occurring Toeplitz and Hankel operators are block operators of a corresponding size and their definition should be obvious.

In the above theorem, we reduced the calculation of the dimensions and pseudoinverse for $T(a)+H(b)$ to those of the singular integral operator $\Phi(A)$. For reasons of symmetry, one should suspect that it can also be done by reduction to the singular integral operator $\Psi(B)$. We will establish the corresponding statement in the following theorem for completeness sake. Of course, the corresponding result should essentially be the same. How the assertions of both of theorems are related with each other will be discussed afterwards.

Theorem 6.2 *Let $a \in G\mathcal{B}^{N\times N}$, $b \in \mathcal{B}^{N\times N}$, and W be given by (114). Introduce the functions*

$$B(t) \;=\; \begin{pmatrix} a(t) & 0 \\ \widetilde{b}(t) & I_N \end{pmatrix} \in \mathcal{B}^{2N\times 2N}, \tag{151}$$

$$G(t) \;=\; \widetilde{B}^{-1}(t)WB(t) \;=\; \begin{pmatrix} \tilde{a}^{-1}(t)\tilde{b}(t) & \tilde{a}^{-1}(t) \\ a(t) - b(t)\tilde{a}^{-1}(t)\tilde{b}(t) & -b(t)\tilde{a}^{-1}(t) \end{pmatrix} \in \mathcal{B}^{2N\times 2N}. \tag{152}$$

If the characteristic pairs of the antisymmetric factorization of $G(t) = \widetilde{B}_+^{-1}D(t)B_+(t)$ are given by (57), then

$$\dim\ker(T(a)+H(b)) \;=\; -\sum_{\varkappa_k<0} \Theta(-\varrho_k, \varkappa_k), \tag{153}$$

$$\dim\ker(T(a)+H(b))^* \;=\; \sum_{\varkappa_k>0} \Theta(-\varrho_k, \varkappa_k). \tag{154}$$

Moreover, if we write

$$B_+^{-1} \;=\; \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \qquad B_+B^{-1} \;=\; \Big(y_1,\ y_2\Big), \tag{155}$$

with $w_1, w_2 \in \mathcal{B}^{N\times 2N}$ *and* $y_1, y_2 \in \mathcal{B}^{2N\times N}$*, then a pseudoinverse of* $T(a)+H(b)$ *is given by*

$$T(w_1)\Big(P - \tfrac{1}{2}H(D^{-1})\Big)\Big(T(y_1)+H(y_2)\Big) \tag{156}$$

Proof. As before, the dimension of the kernel and cokernel of $T(a)+H(b)$ coincides with that of the operator $X = PM(a)P + PM(b)JP + Q$. Now we write

$$PM(a)P + PM(b)JP + Q \;=\; (M(a)P + M(b)JP + Q)(I - QM(a)P - QM(b)JP)$$

Because $Y' = QM(a)P + QM(b)JP$ is nilpotent, the last expression on the right hand side is an invertible operator. Hence we are led to the dimension of the kernel and cokernel of

$$M(a)P + M(b)JP + Q,$$

which coincides with the singular integral operator $\Psi(B)$ where $B(t)$ is given as above. Now the result follows from Theorem 5.4.

Again, a pseudoinverse of $T(a)+H(b)$ is given by $PX^\dagger P$. Since $X = \Psi(B)$Y', it follows that $X^\dagger = (Y')^{-1}(\Psi(B))$. Hence $PX^\dagger P = P(\Psi(B))^\dagger P$ because $P = P(Y')^{-1}$ as can easily be seen. From Theorem 5.4(b) we conclude that $(\Psi(B))^\dagger$ may be given by

$$\mathcal{T}(B_+^{-1})(I - \tfrac{1}{2}\mathcal{H}(D^{-1}W))\Phi(B_+B^{-1})$$

We obtain that this is equal to

$$\begin{aligned}
&\Big(P, JP\Big)M(B_+^{-1})\begin{pmatrix} P \\ PJ \end{pmatrix}\left(I - \frac{1}{2}\Big(P, JP\Big)M(D^{-1})\begin{pmatrix} QJ \\ Q \end{pmatrix}\right)\Big(P, JP\Big)M(B_+B^{-1})\begin{pmatrix} I \\ J \end{pmatrix}\\
&\quad = \quad \Big(P, JP\Big)M(B_+^{-1})\Big(P - \frac{1}{2}PM(D^{-1})JP\Big)M(B_+B^{-1})\begin{pmatrix} I \\ J \end{pmatrix}.
\end{aligned}$$

Hence $PX^\dagger P = P(\Psi(B))^\dagger P$ equals

$$\Big(P, 0\Big)M\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}P\Big(P - \frac{1}{2}H(D^{-1})\Big)PM(y_1, y_2)\begin{pmatrix} P \\ JP \end{pmatrix},$$

which in turn is equal to the operator (156). □

Now we discuss the question of how the statements of the preceding two theorems are related with each other. At first glance, formulas (147) and (148) seem to contradict (153)

and (154) but this is just because the same notation has been used for different factors $D(t)$ with different characteristic pairs. In these theorems we start with the antisymmetric factorizations of certain functions $F(t)$ and $G(t)$:

$$F(t) = A_-(t)D^{(1)}(t)\widetilde{A}_-^{-1}(t), \qquad G(t) = \widetilde{B}_+^{-1}(t)D^{(2)}(t)B_+(t).$$

Assume that the notation of the characteristic pairs is given by $D^{(1)}(t) = \operatorname{diag}(\varrho_i^{(1)} t^{\varkappa_i^{(1)}})$ and $D^{(2)}(t) = \operatorname{diag}(\varrho_i^{(2)} t^{\varkappa_i^{(2)}})$. The functions $F(t)$ and $G(t)$ are given by (146) and (152), from which it follows that

$$G(t) = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} F(t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Note that the constant matrices on the right hand side are the inverses of one another. The last identity is the reason that from an antisymmetric factorization of $F(t)$ one can immediately obtain an antisymmetric factorization of $G(t)$ and vice versa. More precisely, if we are given an antisymmetric factorization of $F(t)$, then an antisymmetric factorization of $G(t)$ is given with the factors

$$B_+(t) = \widetilde{A}_-^{-1}(t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad D^{(2)}(t) = -D^{(1)}(t).$$

This also shows that the construction of an antisymmetric factorization for $F(t)$ and $G(t)$ is essentially the same problem. Moreover, if the characteristic pairs are ordered "appropriately", we may conclude that $(\varrho_k^{(2)}, \varkappa_k^{(2)}) = (-\varrho_k^{(1)}, \varkappa_k^{(1)})$ for all k. This implies that formulas (147) and (148) indeed coincide with (153) and (154).

It is, however, not clear at this point whether formulas (150) and (156) for the pseudoinverse are the same. Observe that pseudoinverses are in general not unique. Of course, if the pseudoinverses are inverses, then they automatically have to be the same.

7 More general singular integral operators

In this section we consider a more general class of singular integral operators. The operators $\Phi(A)$ and $\Psi(A)$ just represent special cases of this class of operators (see (124) and (125)). These more general singular integral operators are operators of the form

$$PM(a_1)P + PM(b_1)JQ + QM(\tilde{c}_1)JP + QM(\tilde{d}_1)Q + \tag{157}$$

$$PM(a_2)JP + PM(b_2)Q + QM(\tilde{c}_2)P + QM(\tilde{d}_2)JQ, \tag{158}$$

where $a_i, b_i, c_i, d_i \in L^\infty(\mathbb{T})^{N\times N}$, $i = 1, 2$. Introducing the functions $A, B \in L^\infty(\mathbb{T})^{2N\times 2N}$ and the constant $W \in \mathbb{C}^{2N\times 2N}$ by

$$A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}, \tag{159}$$

it is easily seen by help of formulas (116) and (117) that the operator (157) equals

$$\mathcal{T}(A) + \mathcal{H}(BW). \tag{160}$$

The following result is an immediate consequence of Proposition 2.1 and Proposition 2.2 in connection with Proposition 5.1. It is also a generalization of Proposition 5.2.

Proposition 7.1 *Let $A, B \in L^\infty(\mathbb{T})^{2N\times 2N}$.*

(a) If $\mathcal{T}(A) + \mathcal{H}(BW)$ is Fredholm, then $A \in G(L^\infty(\mathbb{T})^{2N\times 2N})$.

Let $A, B \in C(\mathbb{T})^{2N\times 2N}$.

(b) $\mathcal{T}(A) + \mathcal{H}(BW)$ is Fredholm if and only if $A \in G(C(\mathbb{T})^{2N\times 2N})$.

Moreover, if this is true, then $\operatorname{ind}(\mathcal{T}(A) + \mathcal{H}(BW)) = -\operatorname{wind}\det A$.

In fact, the mapping Ξ defined in Proposition 5.1 sends the operator $\mathcal{T}(A) + \mathcal{H}(BW)$ into the Toeplitz + Hankel operator $T(A) + H(B)$. In the case where $A \in G(\mathcal{B}^{2N\times 2N})$ and $B \in \mathcal{B}^{2N\times 2N}$, these operators are Fredholm, and formulas for the dimension of the kernel and cokernel can be obtained by help of the results of the previous section.

Theorem 7.2 *Let $A \in G(\mathcal{B}^{2N\times 2N})$ and $B \in \mathcal{B}^{2N\times 2N}$. Introduce the function F by*

$$F = \begin{pmatrix} B\widetilde{A}^{-1} & A - B\widetilde{A}^{-1}\widetilde{B} \\ \widetilde{A}^{-1} & -\widetilde{A}^{-1}\widetilde{B} \end{pmatrix} \in G(\mathcal{B}^{4N\times 4N}). \tag{161}$$

Then F admits an antisymmetric factorization. If the characteristic pairs are denoted by $(\varrho_k, \varkappa_k)$, $k = 1\dots 4N$, then

$$\dim\ker(\mathcal{T}(A) + \mathcal{H}(BW)) = -\sum_{\varkappa_k<0} \Theta(\varrho_k, \varkappa_k), \tag{162}$$

$$\dim\ker(\mathcal{T}(A) + \mathcal{H}(BW))^* = \sum_{\varkappa_k>0} \Theta(\varrho_k, \varkappa_k). \tag{163}$$

It is also possible to establish formulas for the pseudoinverses of $\mathcal{T}(A) + \mathcal{H}(BW)$. We leave these details to the reader.

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